Entanglement Spectrum and a New Principle of Adiabatic Continuity

B. Andrei Bernevig

Department of Physics, Princeton University

Bucharest, December 2009

Collaborators:

Ronny Thomale (Princeton)

Nicolas Regnault (ENS Paris)

A. Sterdyniak (ENS Paris)

BAB and N. Regnault, arXiv:0902.4320, PRL in press
BAB and F.D.M. Haldane, Phys. Rev. Lett. 100, 246802 (2008)
R. Thomale, A. Sterdyniak, N. Regnault, and BAB (2009)

Contents:

• Topological Order VS Normal Order

•Detailed introduction to the Fractional Quantum Hall effect, fractional statistics and edge excitations.

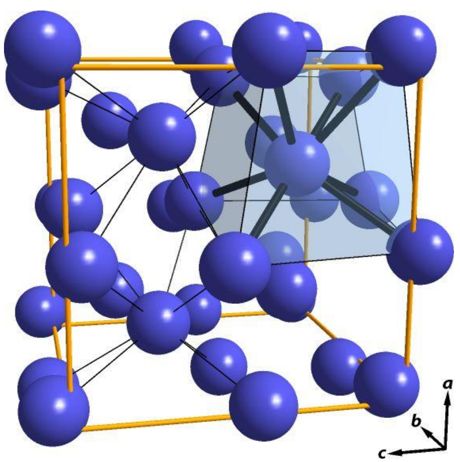
• How to determine topological order in generic ground-state wavefunction of a realistic (say Coulomb) Hamiltonian?

• Entanglement spectrum in the conformal limit

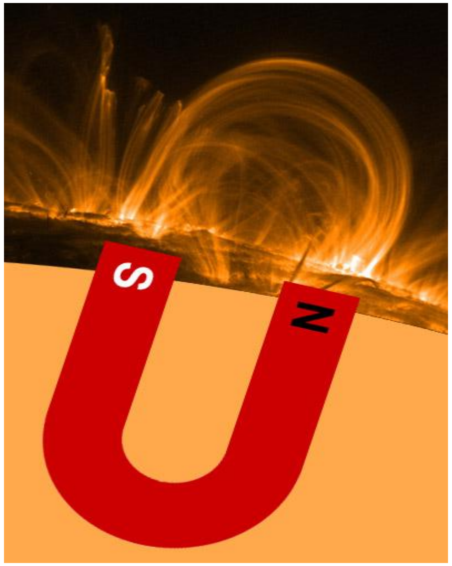
• New principle of adiabatic continuity.

Most States of Matter are defined by Broken Symmetry

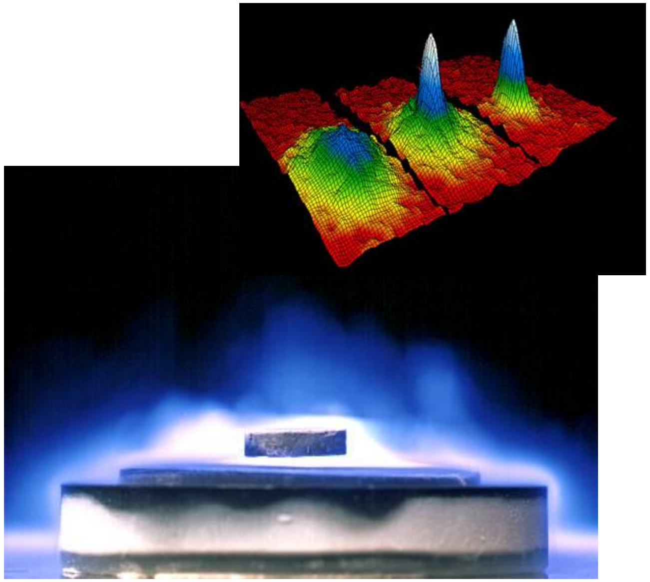
Same constituent forces, different macroscopic states of matter: liquid, vapor, crystal.
Broken Symmetry: differentiates different states of matter.
Condensed matter: finding different states of matter.



Crystal:Broken translational symmetry



Magnet: Broken spin rotational symmetry

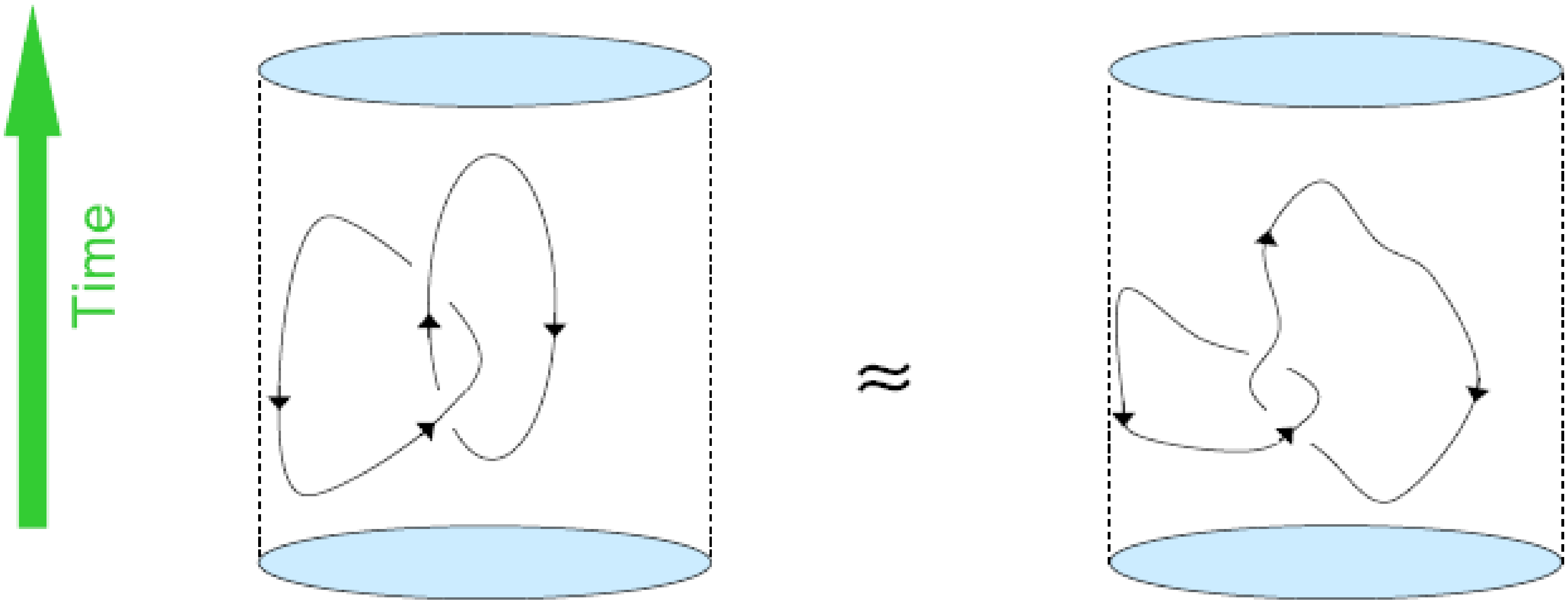


Superconductor: Broken gauge symmetry

Topological Phase: If Described by a Topological QFT

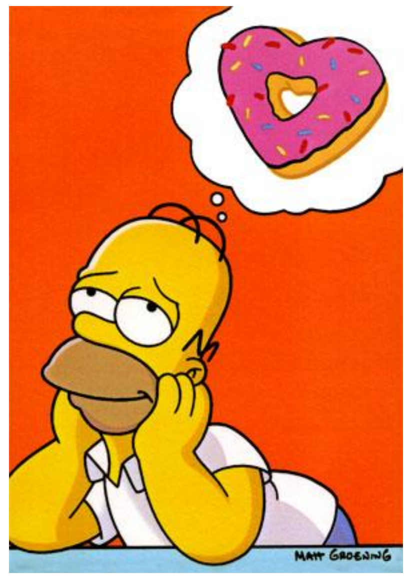
TQFT = QFT where amplitudes depend only on the topology of the process

$$\mathcal{Z} = \int \mathcal{D}\{configs\}\; e^{iS(config)}$$

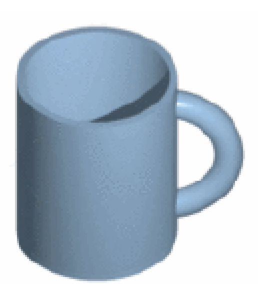


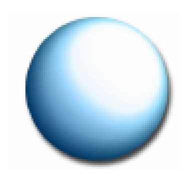
- Def#2: A phase of matter where, at long distance and low energy, observables (correlators) are invariant under smooth deformation of space-time.

A Topological Phase Feels the Manifold on Which it Sits

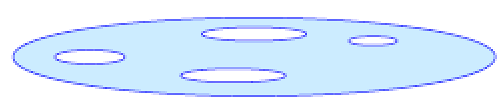


- Different ground-state degeneracy depending on genus of manifold
- This definition applies only to bulk gapped systems
- State “fits” differently on sphere and torus (property called “shift” in Fractional Quantum Hall systems)

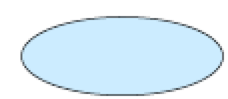




Sphere:

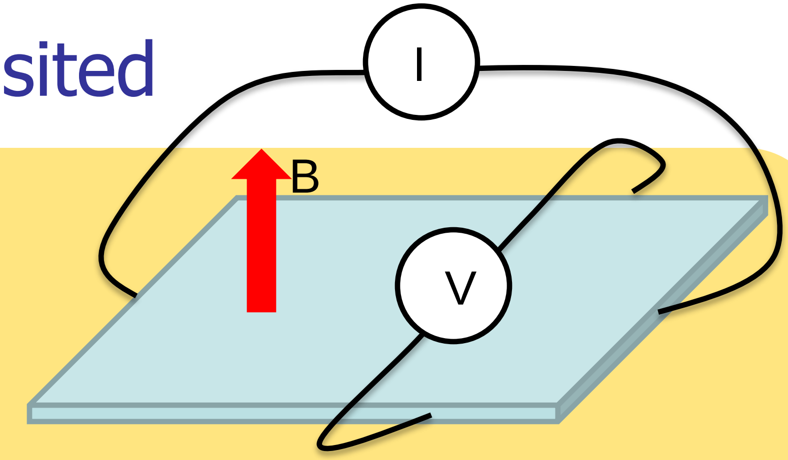


Punctured Disk:



Disk:

The Quantum Hall Effect Revisited



- High Magnetic Fields (20-30 T)
- 2D Electron Gas (disorder!)
- Low Temperatures (0.1-10 K)
- Two kinds of Quantum Hall: Integer and Fractional. Fractional much more interesting. Start with Integer FQH:

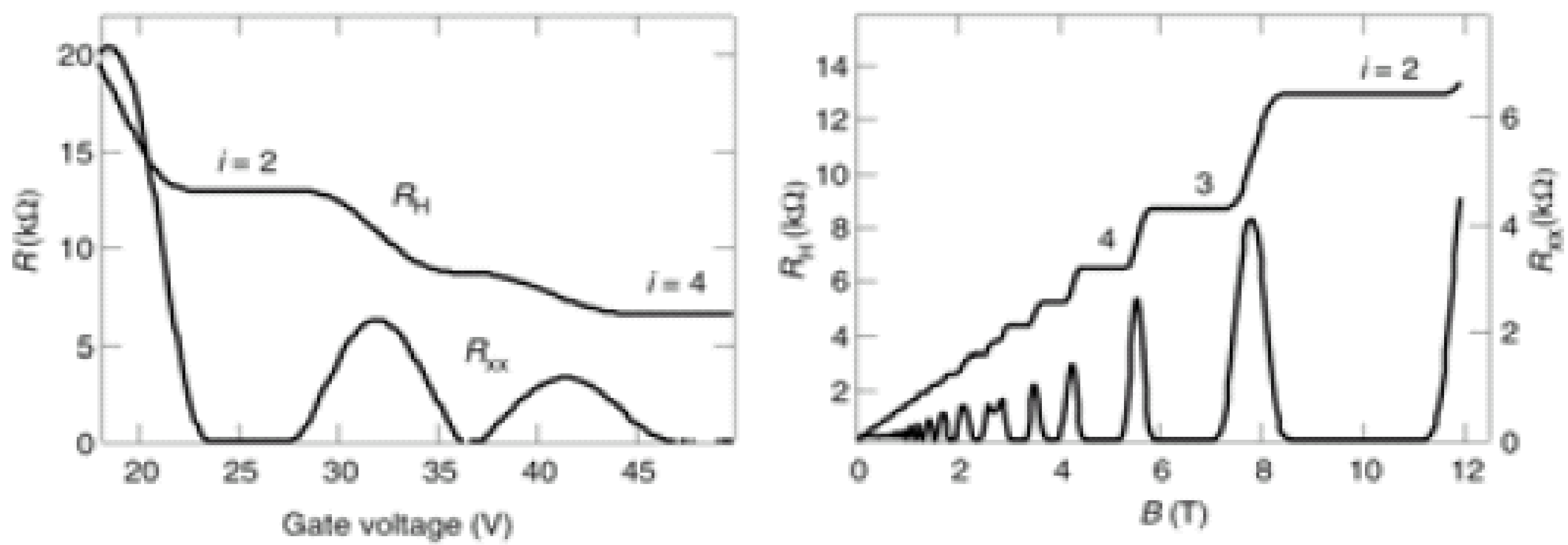


Figure 1. Experimental measurements of the Hall resistance R_H and of the longitudinal resistance R_{xx} for a Si-MOSFET ($B = 13.8$ T) and a GaAs/AlGaAs heterostructure at a temperature of 0.3 K.

The Integer Quantum Hall Effect: Landau Levels in Symmetric Gauge

If $\vec{B} = (0,0,B)$, then $\vec{A} = \frac{1}{2}\vec{r}\times\vec{B} = \frac{B}{2}(-y,x,0)$

$$\frac{\vec{p}^2}{2m} \rightarrow \frac{\left(\vec{p}+e\vec{A}\right)^2}{2m} = \frac{p_x^2+p_y^2}{2m} + \frac{e^2B^2\left(x^2+y^2\right)}{8m} + \boxed{\frac{eB}{2m}\left(xp_y - yp_x\right)}$$

$$E = (2n+1)\hbar\omega_L = E = \left(n+\frac{1}{2}\right)\hbar\omega_c$$

Lowest Landau Level (n=0) eigenstates:

$$\psi_{\text{single particle}}(z) = \frac{1}{\sqrt{2^m m!}} z^m \exp(-|z|^2/(4l_b^2))$$

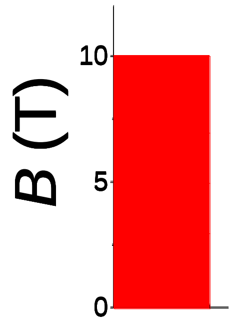
$$m = 1, \ldots, N_\Phi \qquad \ell_B = \sqrt{\frac{\hbar c}{eB}}$$

Integer Quantum Hall is just single-electron physics (+localization)

The Integer Quantum Hall Effect: Landau Levels

B=0:

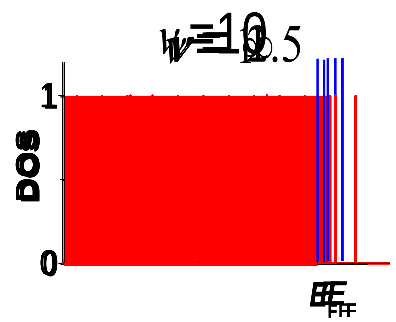
2D DOS is constant



B>0:

DOS becomes series of d-functions:

Landau Levels



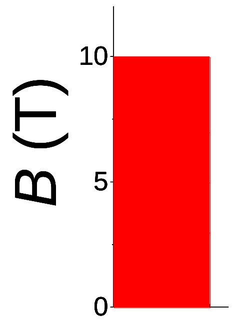
energy separation:

$$\hbar\omega_c = \frac{\hbar eB}{m^*}$$

The Quantum Hall Effect: Landau Levels

B=0:

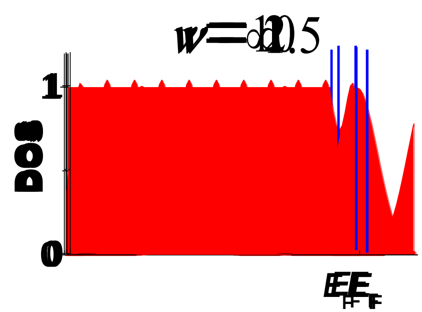
2D DOS is constant



broadening due to disorder

B>0:

DOS becomes series of d-functions: Landau Levels



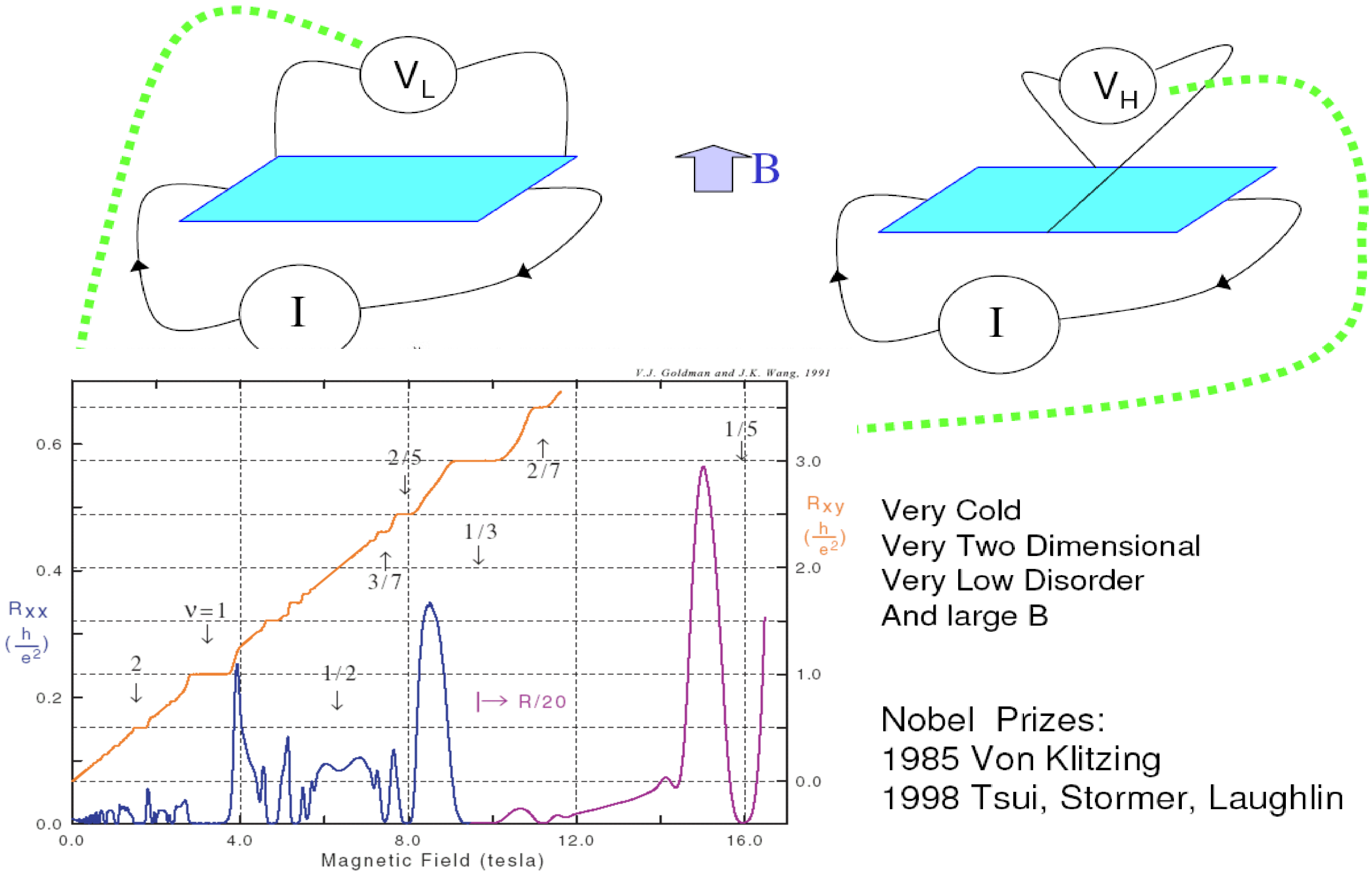
2D states (B=0,T=0) are localized, but extended states in center of Landau Levels

energy separation:

$$\hbar\omega_c = \frac{\hbar eB}{m^*}$$

Fractional Quantum Hall: First Example of Topological Phase

• In 1982, it was observed that a Quantum Hall state appears at FRACTIONAL filling, i.e. when LESS than one full Landau Level is occupied. Interacting!



Very Cold
Very Two Dimensional
Very Low Disorder
And large B

Nobel Prizes:
1985 Von Klitzing
1998 Tsui, Stormer, Laughlin

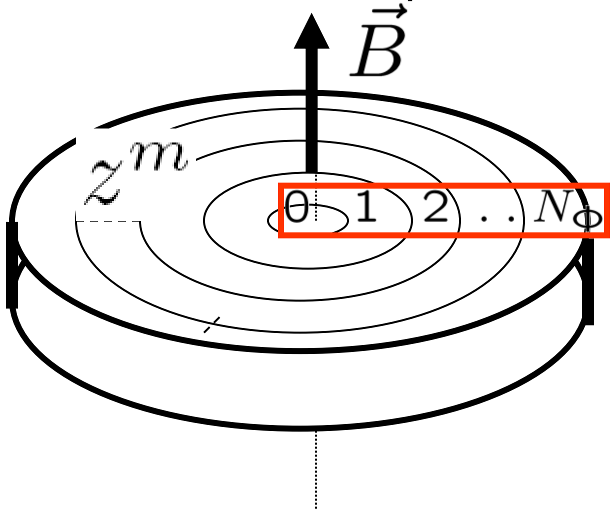
The Laughlin's Model State is the Simplest Example of a FQH State

- First (most) FQH states observed are in the lowest Landau Level (LLL)

- The non-interacting single particle wavefunctions in the LLL are holomorphic

$$\psi_{\text{single particle}}(z) \sim z^m$$

$$m = 1, ..., N_\Phi$$



- The fully filled Landau level is simply a SINGLE Slater determinant!

$$\psi_{\text{filled LL}}(z_1, ..., z_N) \sim \det z_i^j = \prod_{i<j}(z_i - z_j), \qquad i = 1, ..., N; \quad j = 0, ..., N_\phi$$

$N_\Phi = N - 1$ for $1st$ Landau Level, which is at filling $\nu = 1 = \frac{N}{N_\phi}$

The Laughlin State is the Simplest Example of a FQH State

The first FQH state observed was at filling factor $\nu = 1/3$.
Laughlin proposed a Variational wavefunction to describe this state:

$$\psi_{\text{Laughlin}}(z_1, \ldots, z_N) = \prod_{i<j}(z_i - z_j)^3$$

The Laughlin State, although not an exact ground-state of a Coulomb Hamiltonian, has properties that make it an extremely good variational ground state:

- Translationally invariant
- Has very small probability when two particles are on top of each other (very good for coulomb)

- It is now abundantly clear that the coulomb ground-state of the FQH system at filling 1/3, although dependent on microscopic details of the sample, is in the same UNIVERSALITY class as the Laughlin state

- In finite size numerics, Haldane and Rezayi were able to adiabatically continue between Coulomb ground-state and Laughlin state.

Quasiparticle excitations above the Laughlin State Obey Fractional Statistics!

• Bulk excitations have a thermodynamic limit finite energy GAP

•Neither bosons, nor fermions, but ANYons . Only possible in system in Low dimensions (<3), and with hard-core repulsion, as well as with Time-Reversal and Parity breaking. FQH has all of these.

• For an excitation at position A and one at B far away from A, with electrons at position 1,2,...,N, the wavefunction is:

$$\psi(z_A, z_B; z_1, z_2, ..., z_N) = \frac{1}{(z_A - z_B)^{\frac{1}{3}}} \prod_{i=1}^{N} (z_A - z_i)(z_B - z_i) \prod_{i<j} (z_i - z_j)^3$$

• Upon rotating B around A by 180 (interchange), we get a FRACTIONAL phase factor instead of the usual bosonic/fermionic 0 or 1

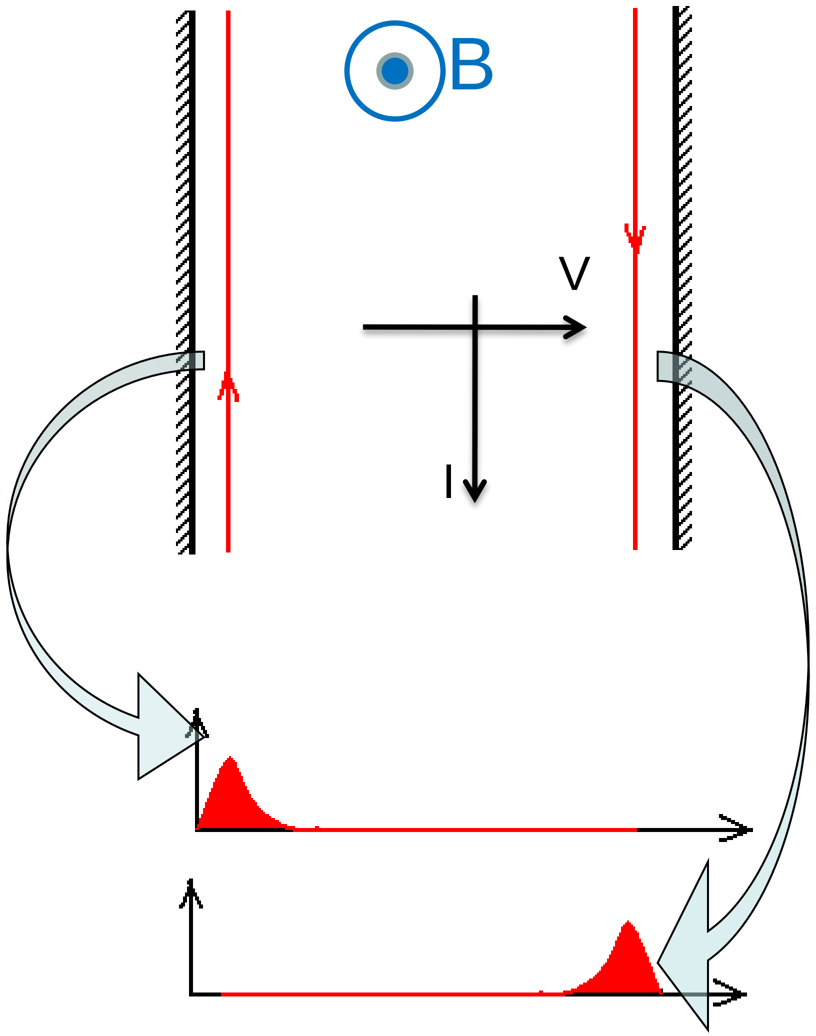
The FQH state is also the first example of a holographic principle: bulk-edge duality.

- If the bulk is gapped (bulk insulator), how does electric conduction take place?

- The bulk is GAPPED but the edge excitations are gapless

- Conduction takes place exclusively through the edge states

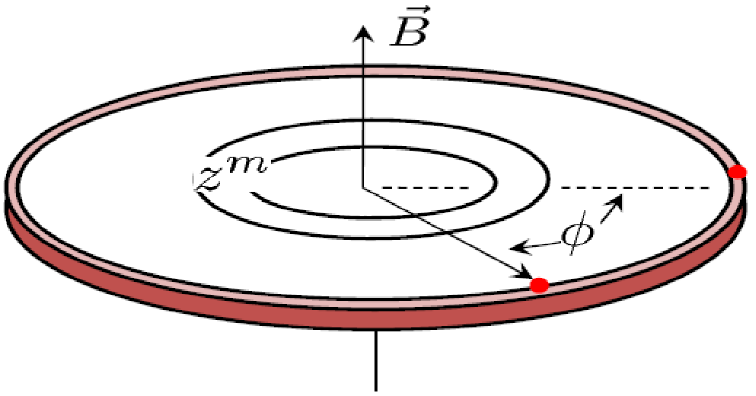
- The chirality of the edge states is given by the applied magnetic field



Counting of edge excitations in angular momentum sectors for Laughlin state has the form: 1,1,2,3,5,7….

$$\psi_L = \prod_{i<j}^{N} (z_i - z_j)^r \qquad |0\rangle$$

"Fermi Sea" on the edge



- Zeroes of the wavefunction sit on the particles
- Excitations on the edge:

$$\psi_{L=1} = \sum_{i=1}^{N} z_i \psi_L \qquad \phi_1|0\rangle$$

$$\psi_{L=2} = \sum_{i\neq j}^{N} z_i^2 \psi_L \qquad \phi_2|0\rangle$$

$$\psi_{L=2} = \sum_{i\neq j}^{N} z_i z_j \psi_L \qquad \phi_1\phi_1|0\rangle$$

$$\psi_{L=3} = \sum_{i=1}^{N} z_i^3 \psi_L \qquad \phi_3|0\rangle$$

$$\psi_{L=3} = \sum_{i\neq j}^{N} z_i z_j^2 \psi_L \qquad \phi_1\phi_2|0\rangle$$

$$\psi_{L=3} = \sum_{i\neq j\neq k}^{N} z_i z_j z_k \psi_L \qquad \phi_1\phi_1\phi_1|0\rangle$$

- Counting of modes at ang. mom. L = partitions of L

COUNTING of U(1) BOSON

•MANY other FQH states besides the one at the Laughlin filling 1/3 have been observed.

•For some of them, we have variational wavefunctions and can determine their properties. For others we don’t

•Some are predicted to have non-abelian statistics!

•How do we know which state happens in the realistic sample when we do NOT yet have a variational wavefunction to compare it with.

•Even when we have a variational wavefunction, the main method of comparison is wavefunction overlap, which is flawed.

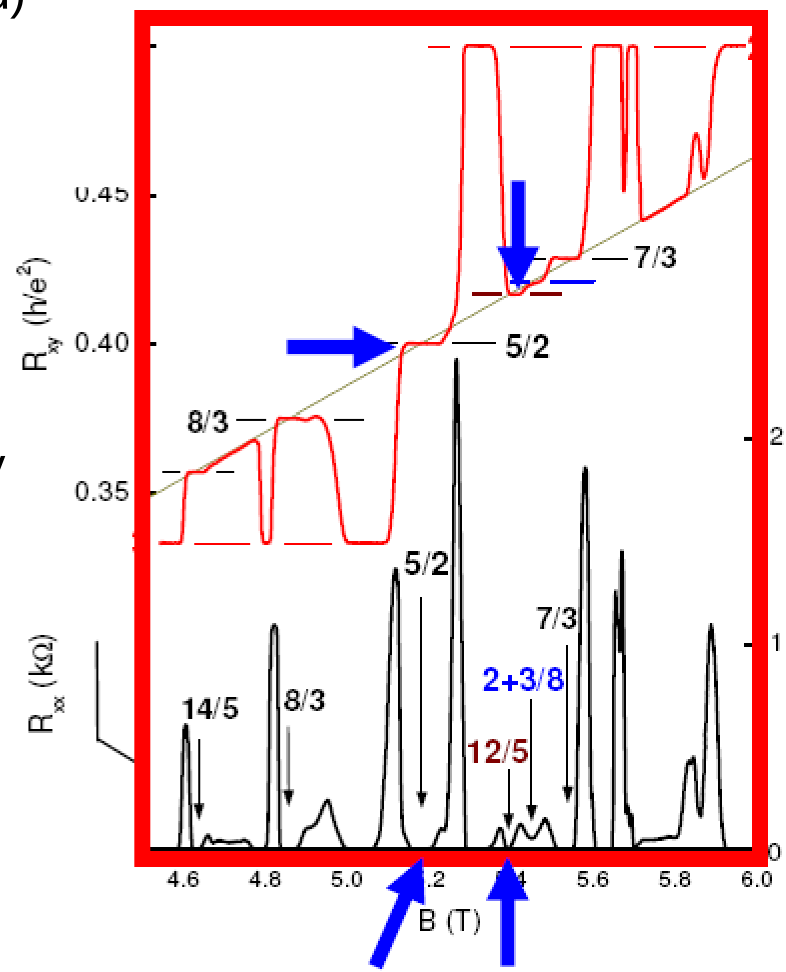
Non-Abelian Top Phases exist! 5/2 Fractional Quantum Hall State

Mobility 31 million cm^2/ V-sec
T = 9 mK (Xia et al, PRL 2004)

- Numerics seems to favor more exotic, non-abelian states at 5/2 (Moore-Read) and 12/5 (Maybe Read-Rezayi)

$5/2 \approx SU(2)_2$
$12/5 \approx SU(2)_3$

Maybe, experiments/numerics underway

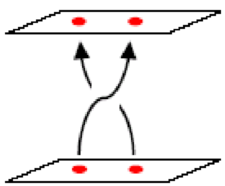


- Proposals to realize similar physics in:

- Other quantum Hall states
- Chiral p-wave superconductors (Sr_2RuO_4),
- Superfluid He-3 films
- Cold Atomic Gases (Rotating or not)
- Atom/Ion Lattices
- Josephson Junction Arrays,
- Bismuth-Antimonide-Super Junctions..
- others?

FQH Could Be Used For Topological Quantum Computing

Quantum computing connection: Freedman, Kitaev,...



Abelian Representation of Braid Group

W = Topological Winding Number of Braid

$$\Psi_f = e^{iW\alpha}\,\Psi_i$$

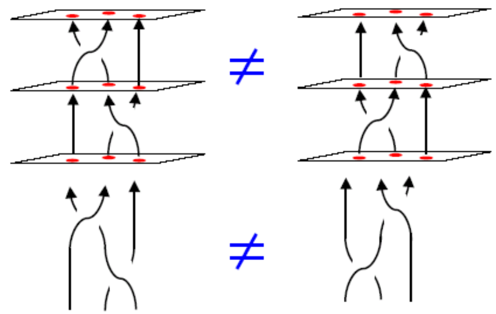
α = "Statistical Angle"

Bosons: $\alpha=0$

Fermions: $\alpha=\pi$

Anyons: other α

Statistical angle =pi/3 for 1/3 Laughlin state quasiholes



Non-Abelian Representation of Braid Group

Suppose 2 Degenerate Orthogonal States $|\psi_1\rangle$, $|\psi_2\rangle$

Vector Represents State

$$\mathbf{\Psi}_i = a_1|\psi_1\rangle + a_2|\psi_2\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\mathbf{\Psi}_f = \tilde{a}_1|\psi_1\rangle + \tilde{a}_2|\psi_2\rangle = \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix}$$

$$\begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix} = U \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Matrix (U)

Laughlin (Abelian) and Moore-Read (Non-Abelian) FQH States

$$\psi_L = \prod_{i<j}^{N} (z_i - z_j)^r$$

- Zeroes of the wavefunction sit on the particles

  - Non-abelian State:

$$\Psi^0_{MR}(z_1, \ldots, z_N) = Pf\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^2$$

- Zeroes of the wf NOT on the particles; vanishes for 3 particles together.

- These "clustering", zero conditions define Hamiltonians for which Laughlin and Moore-Read States are zero energy eigenstates

- FQH states, at whatever filling, share some common general properties, the most fundamental of which are 1: fractional abelian or nonabelian statistics and 2: the bulk-edge duality

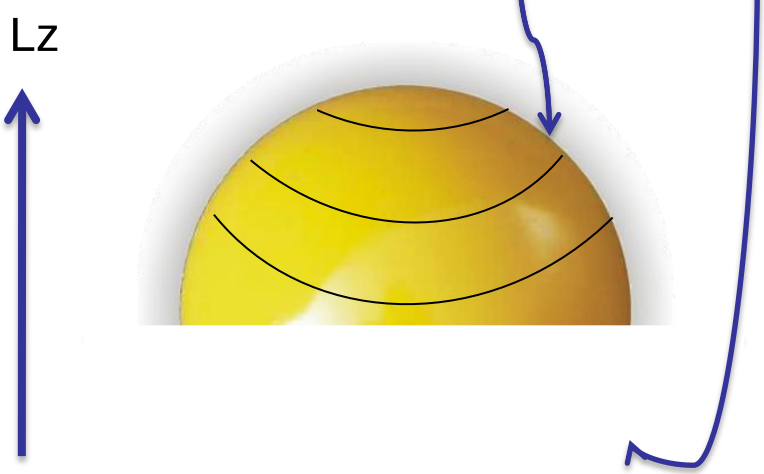
- The number of edge state excitations at each angular momentum is an imprint of the state

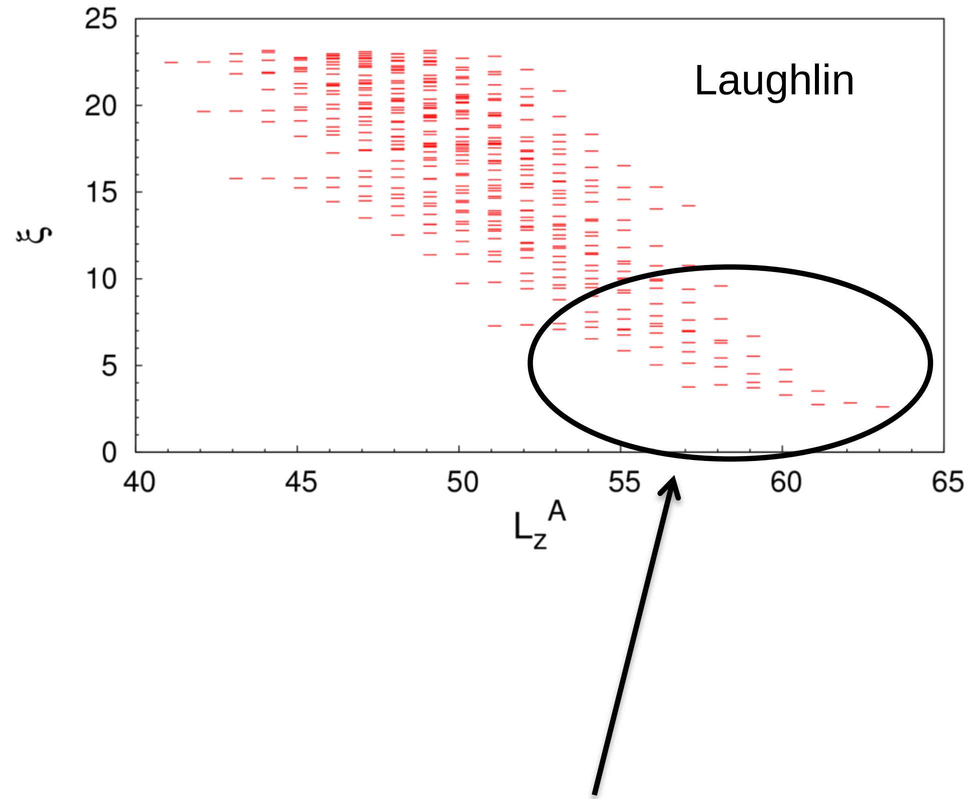
- How to get this out of a realistic Hamiltonian of a sample

Look at Topological Entanglement Spectrum, Numerically, Model States Exhibits Far Fewer Levels Than They Should

Hui and Haldane, 2008;
Regault, BAB, Haldane 2009;
BAB and Regnault, 2009;

$$|\psi\rangle = \sum_{\alpha} e^{-\xi_\alpha/2} |\psi_{N_\alpha}\rangle \otimes |\psi_{S_\alpha}\rangle$$





Cutting an N=14 particle state in half, since Landau orbitals are localized in space, the entanglement matrix should be order 7! X 7! with 7! eigenvalues. This happens in the Coulomb state, but NOT in model Laughlin

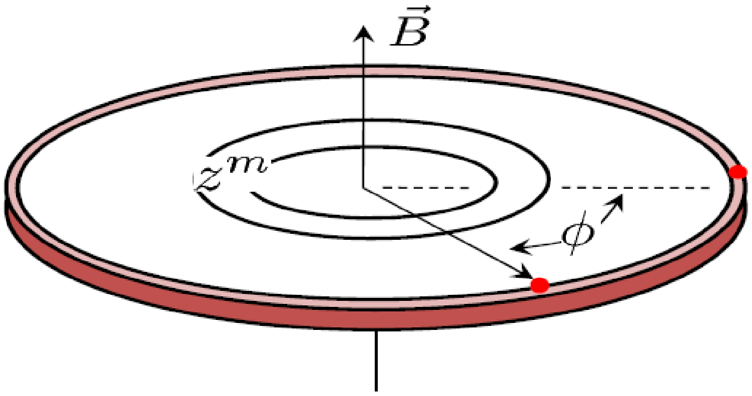
Observe a remarkable counting structure of the eigenvalues of the entanglement spectrum: 1,1,2,3,5, etc

DIRECTLY FROM GROUND-STATE!

The counting of the entanglement levels of the entanglement spectrum for Laughlin corresponds to the number of Edge States! Much Better at Characterizing Topological Order

$$\psi_L = \prod_{i<j}^{N} (z_i - z_j)^r \qquad |0\rangle$$

"Fermi Sea" on the edge



- Zeroes of the wavefunction sit on the particles
- Excitations on the edge:

$$\psi_{L=1} = \sum_{i=1}^{N} z_i \psi_L \qquad \phi_1|0\rangle$$

$$\psi_{L=2} = \sum_{i\neq j}^{N} z_i^2 \psi_L \qquad \phi_2|0\rangle$$

$$\psi_{L=2} = \sum_{i\neq j}^{N} z_i z_j \psi_L \qquad \phi_1\phi_1|0\rangle$$

$$\psi_{L=3} = \sum_{i=1}^{N} z_i^3 \psi_L \qquad \phi_3|0\rangle$$

$$\psi_{L=3} = \sum_{i\neq j}^{N} z_i z_j^2 \psi_L \qquad \phi_1\phi_2|0\rangle$$

$$\psi_{L=3} = \sum_{i\neq j\neq k}^{N} z_i z_j z_k \psi_L \qquad \phi_1\phi_1\phi_1|0\rangle$$

- Counting of modes at ang. mom. L = partitions of L

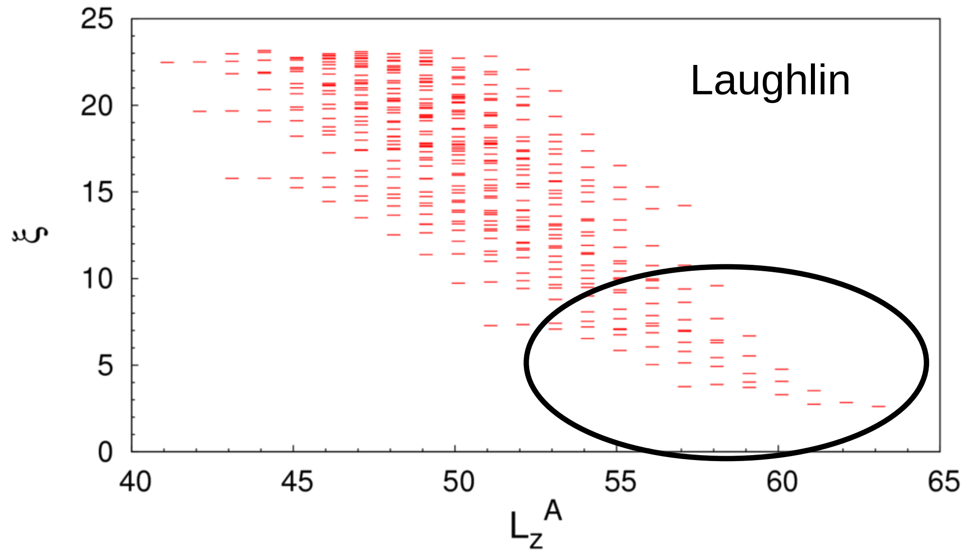
COUNTING of U(1) BOSON

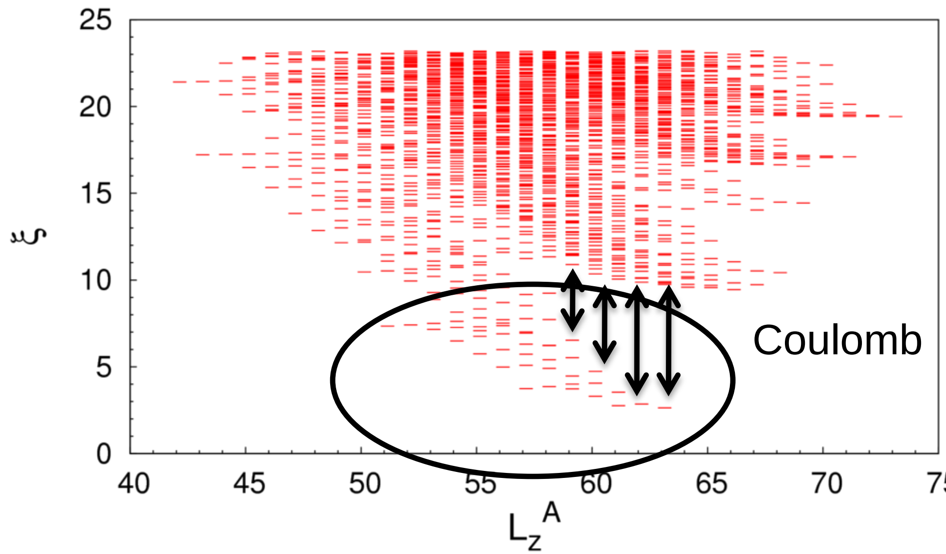
The generic state (diagonalization of a realistic Hamiltonian at filling 1/3) shows many levels, but some of them are separated by an entanglement gap from the Laughlin low entanglement energy levels

Hui and Haldane, 2008;
Regault, BAB, Haldane 2009;
BAB and Regnault, 2009;

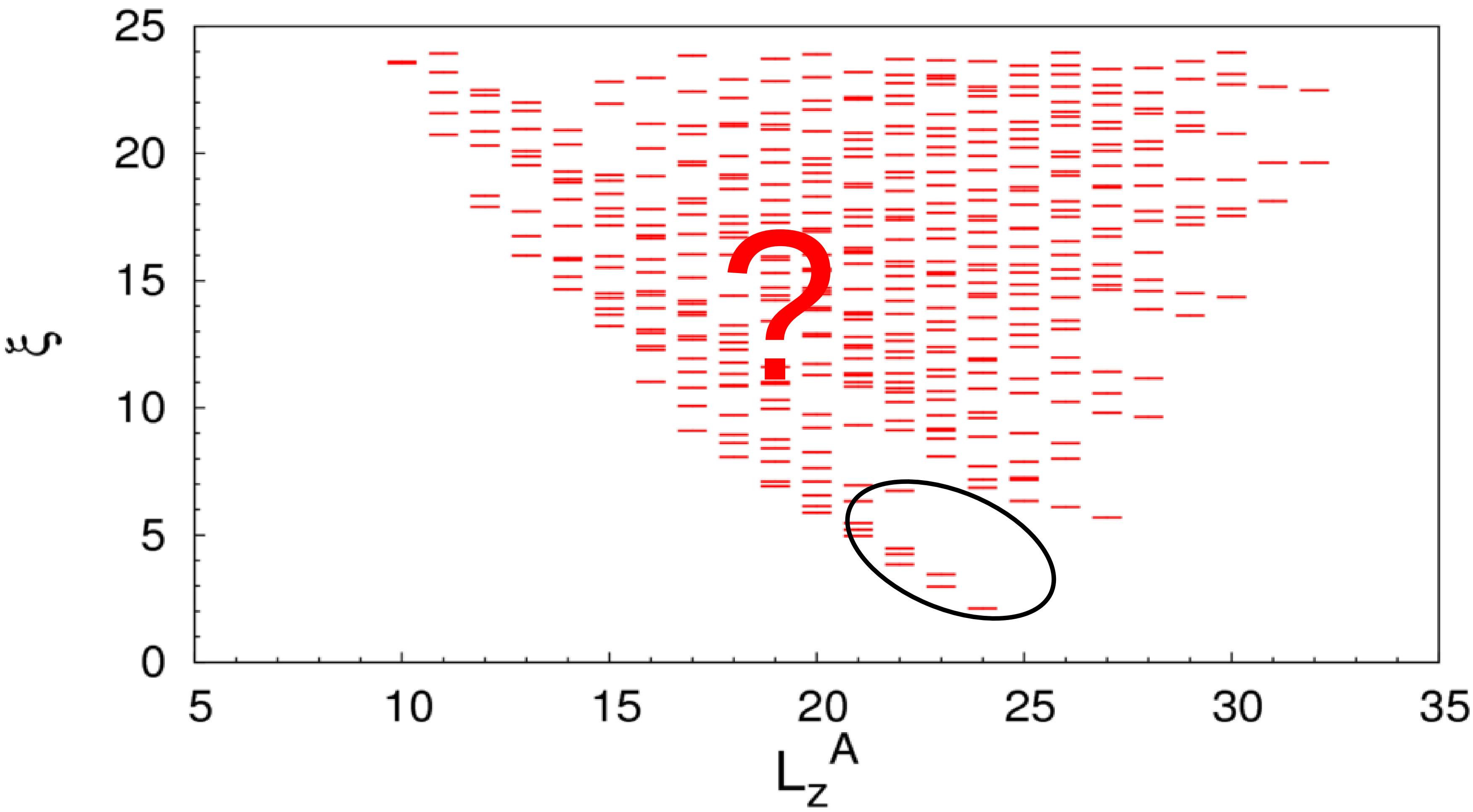
However, entanglement gap might be small, impossible to differentiate between states (see slide)

How do we properly indentify the entanglement gap





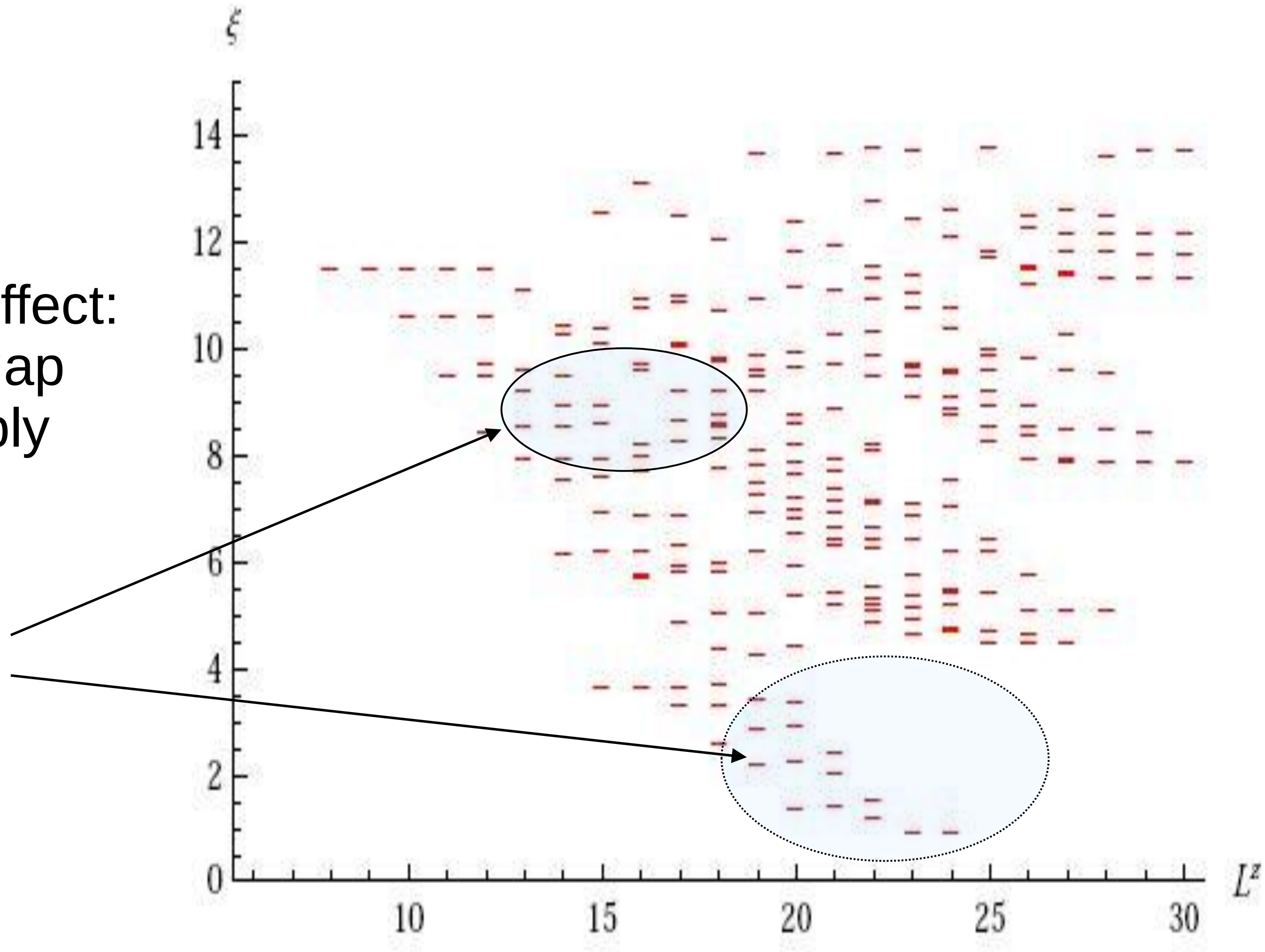
Most Realistic States Show Entanglement Gap Only For a Few Values



Short-Range Interaction (Coulomb) for at filling 5/2 for N=14. Has entanglement gap for only the first 2-3 values. Hard to distinguish the universality class of this state, since many states could have counting 1,2, 4.

Now Come Back to Question of How to Define Entanglement Gap, and How to See with High Sensitivity, Top Order in a generic Ground-State

Quantum Hall effect: Entanglement gap cannot be reliably defined on the sphere.



Cure for the Quantum Hall state: Conformal limit basis, remove any information about magnetic length scales

Normalization of slater determinants on the plane or sphere contains nonuniversal information such as magnetic length

$$\psi_{\text{single particle}}(z) = \frac{1}{\sqrt{2^m m!}} z^m \exp(-|z|^2/(4l_b^2))$$

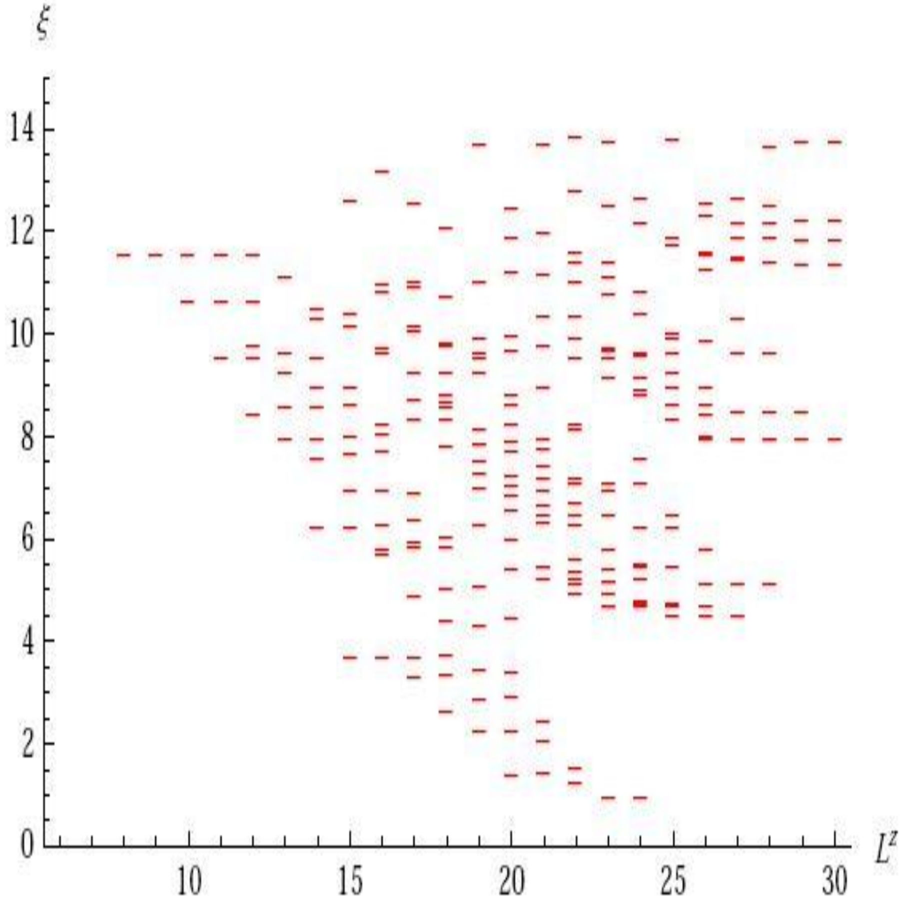
$$m = 1, ..., N_\Phi \qquad \ell_B = \sqrt{\frac{\hbar c}{eB}}$$

Normalization factors on the sphere and plane are orbital-dependent (curvature), we want to remove them.

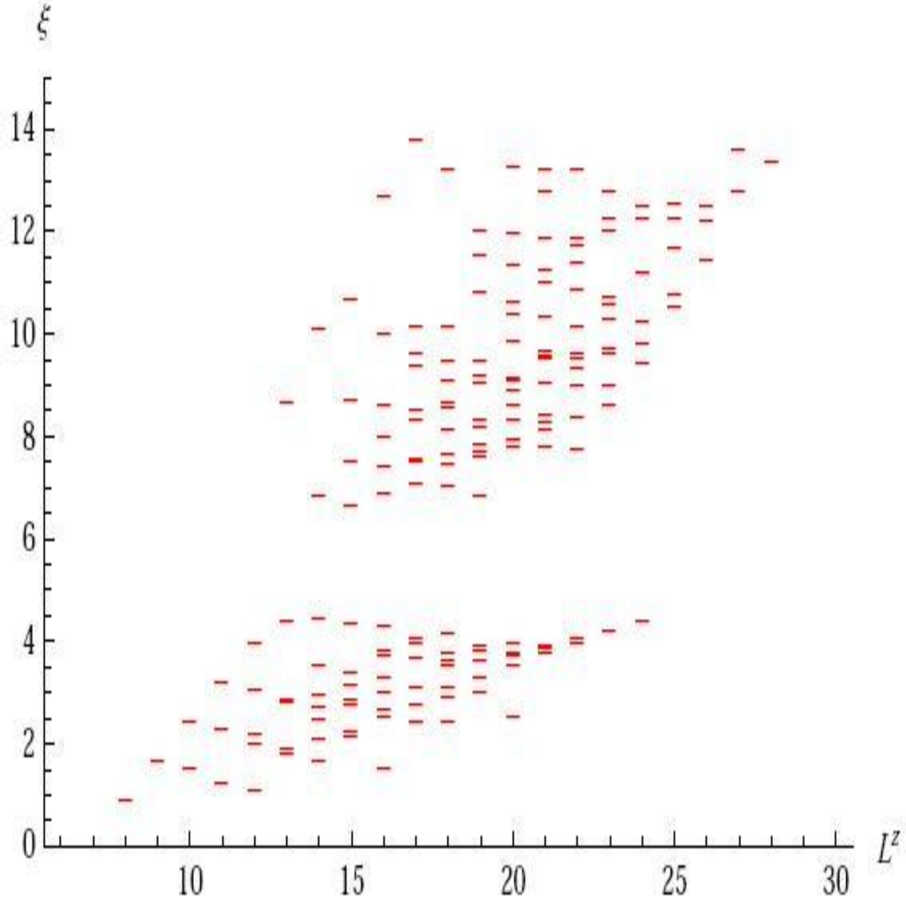
$\longrightarrow$ Also for bosons, remove (bosonic) multiplicities

1/3 Laughlin state

sphere normalization

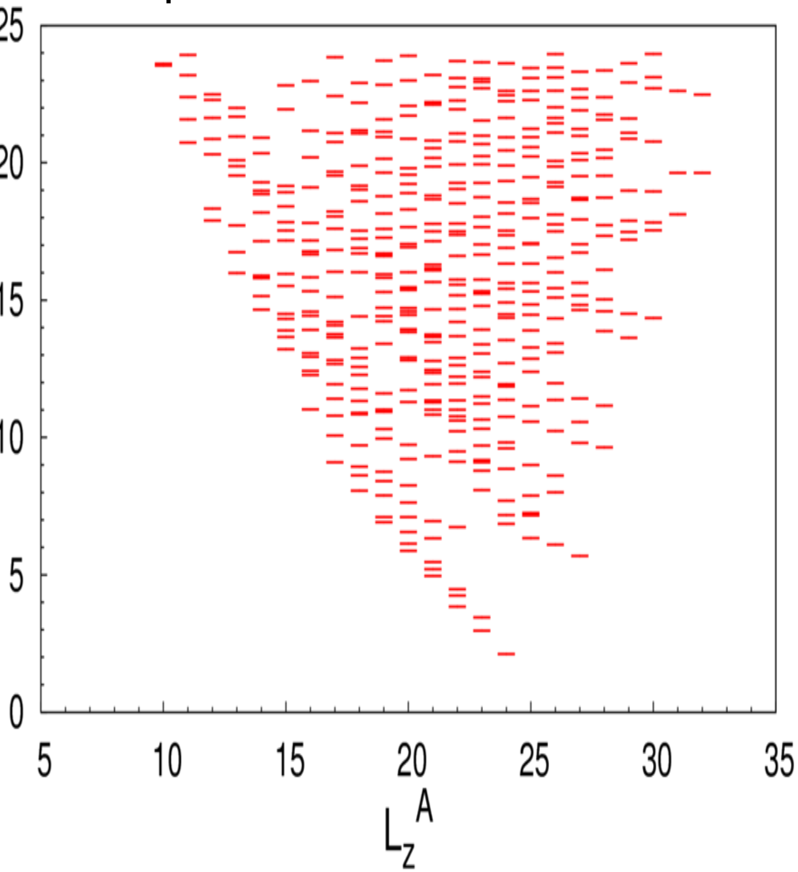


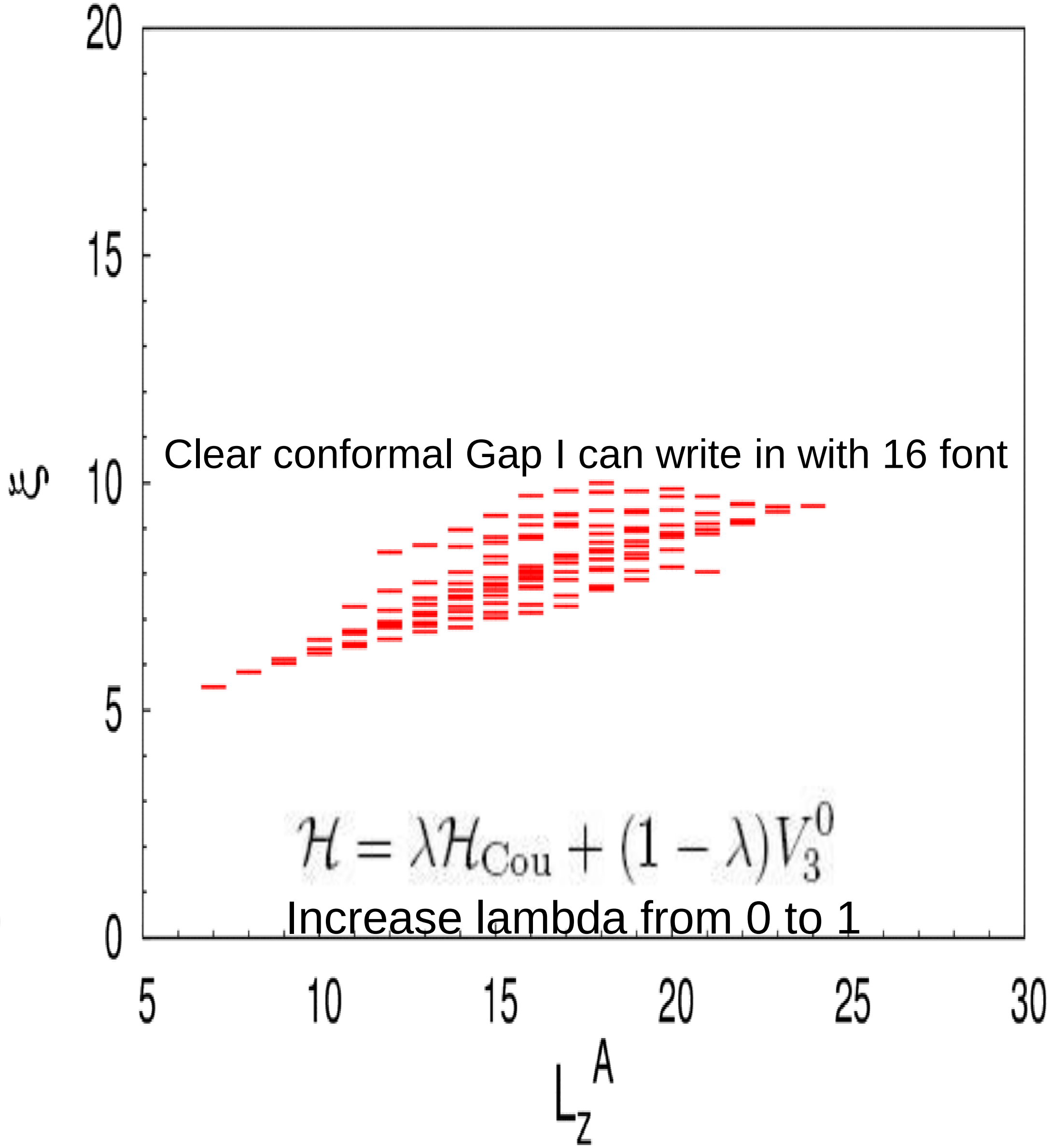
conformal limit



Conformal limit basis allows to define a clean entanglement gap and to resolve topological adiabacity connection for moore-read state

Sphere Calculation





Energy Adiabaticity VS Entanglement Adiabaticity

Adiabatic theorem, (energy): two ground-states of two different Hamiltonians H_1, H_2 belong to the same universality class if there is a path in the parameter space (say $(1-\lambda)H_1 + \lambda H_2$, with $\lambda \in [0,1]$ such that the energy gap doesn't collapse

Requires knowledge of FULL spectrum, ground-states AND excited states of the two hamiltonians..

Adiabatic theorem, (entanglement): two ground-states of two different Hamiltonians H_1, H_2 belong to the same universality class if there is a path in the parameter space (say $(1-\lambda)H_1 + \lambda H_2$, with $\lambda \in [0,1]$ such that the entanglement gap doesn't collapse

Requires knowledge ONLY of the ground-states. Extra bonus for free: the counting gives you the universality class of the state

Conclusions:

- Topological phases/FQH still hold many interesting mysteries

- Product rule unravels the most important one: most of the weight of a model FQH state is held in the product rule configurations

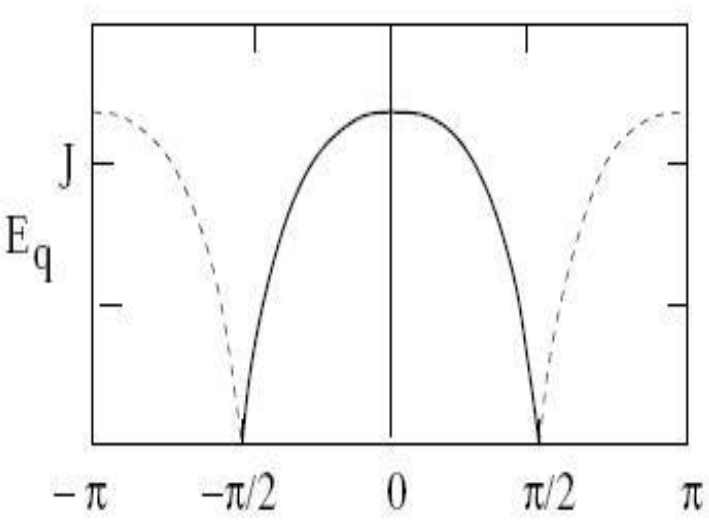
- Explains part of the entanglement spectrum

- Suggests the “conformal limit” of the Haldane entanglement spectrum

- In this limit, easy to define entanglement gap and literally “SEE” the universality of the state

- Ent spectrum sees the low energy theory: for gapped sytems, these are edge states.

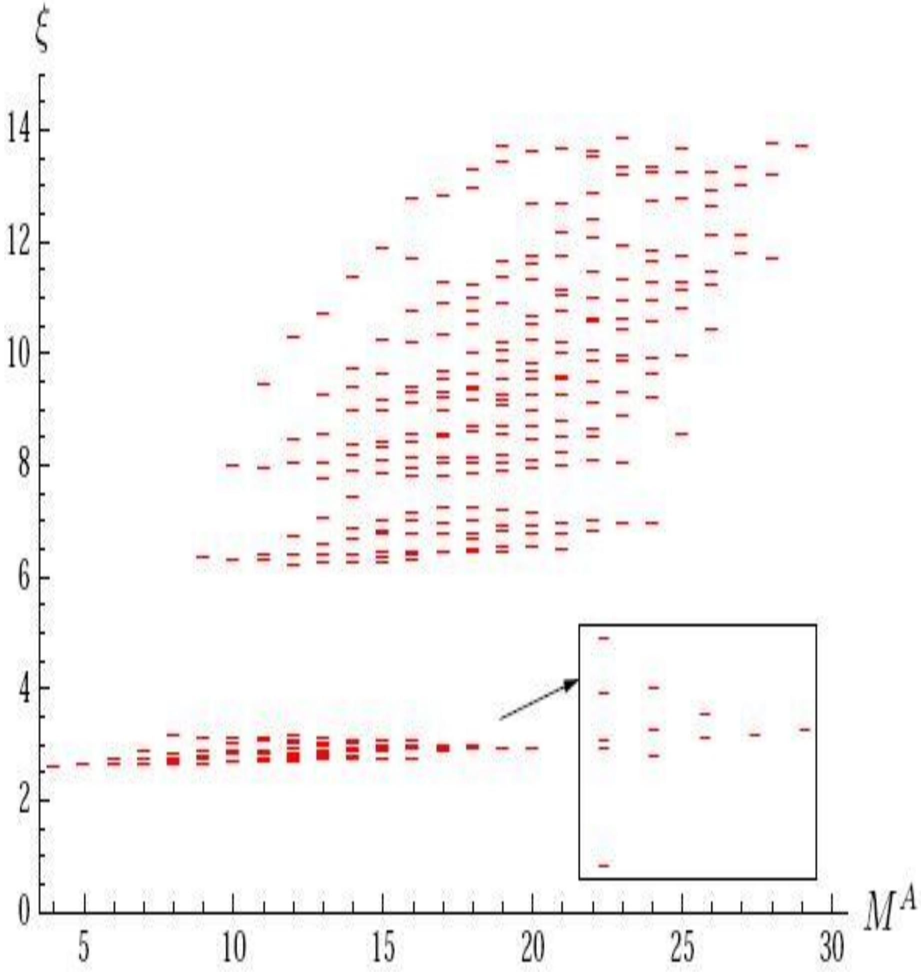
Origin of low energy state branch: Spinon modes



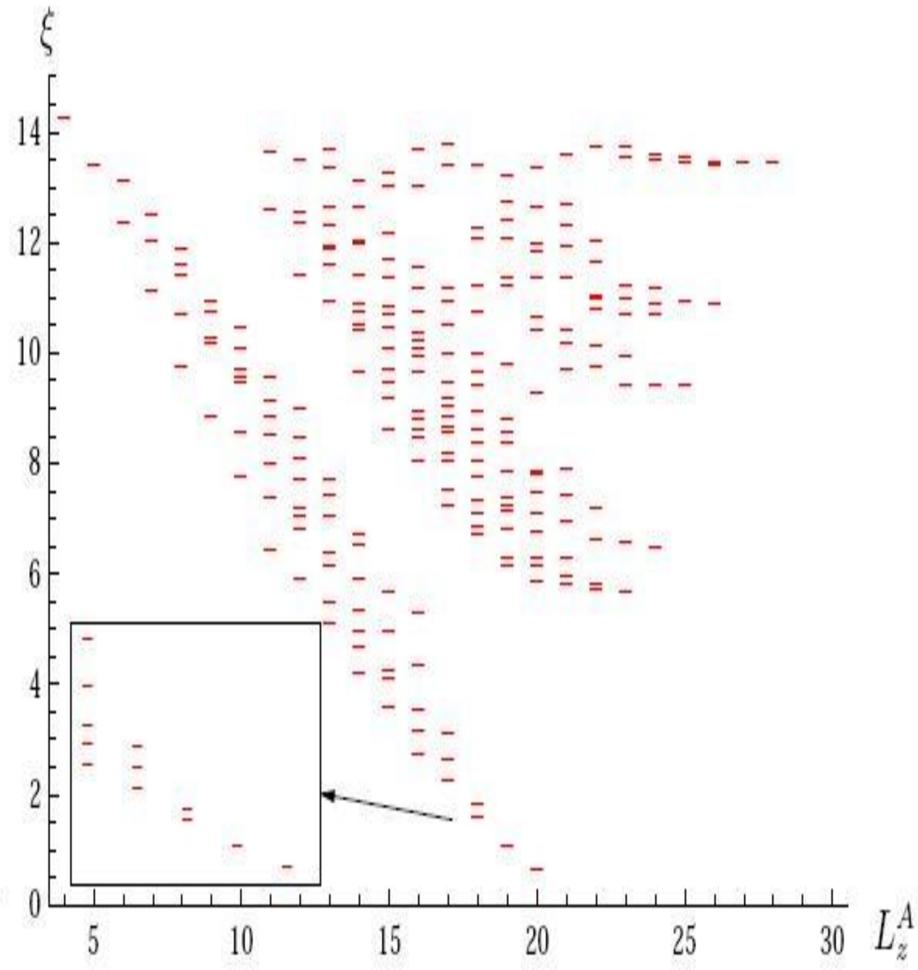
Momentum cut resolves the elementary gapless bulk excitations as the dominant entanglement levels of the Heisenberg state

Comparison to ES for Quantum Hall states

ES of Heisenberg state in momentum orbital basis

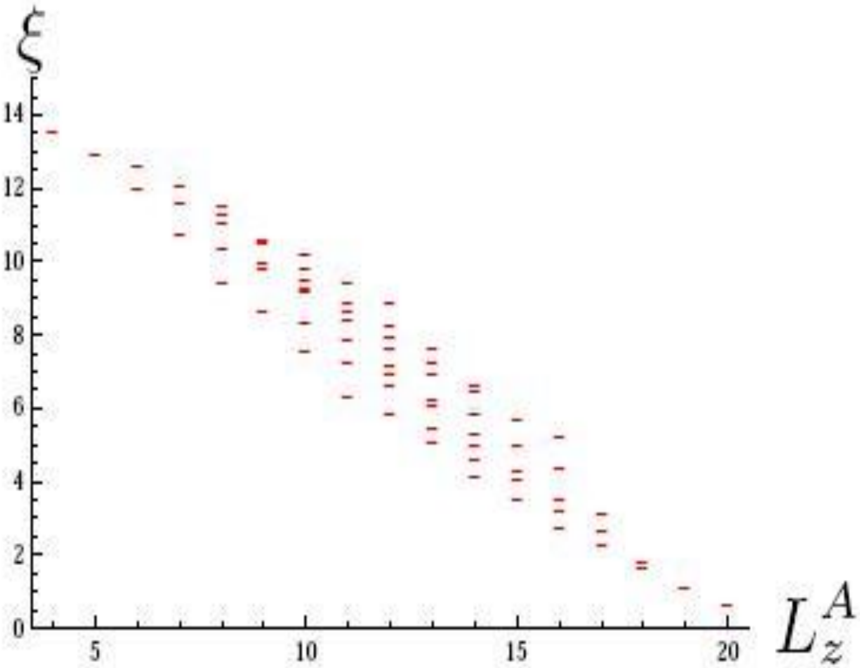


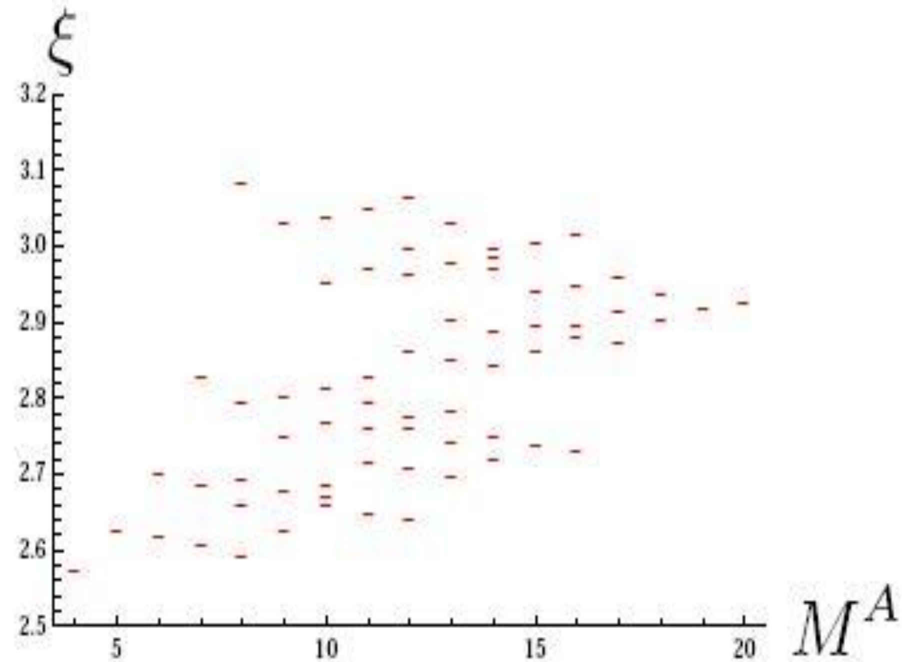
ES of ½ Coulomb state on quantum Hall sphere



Connection between spin chains and Quantum Hall:
the ½ Laughlin state is the Fourier transform of the Haldane-Shastry spin chain state.

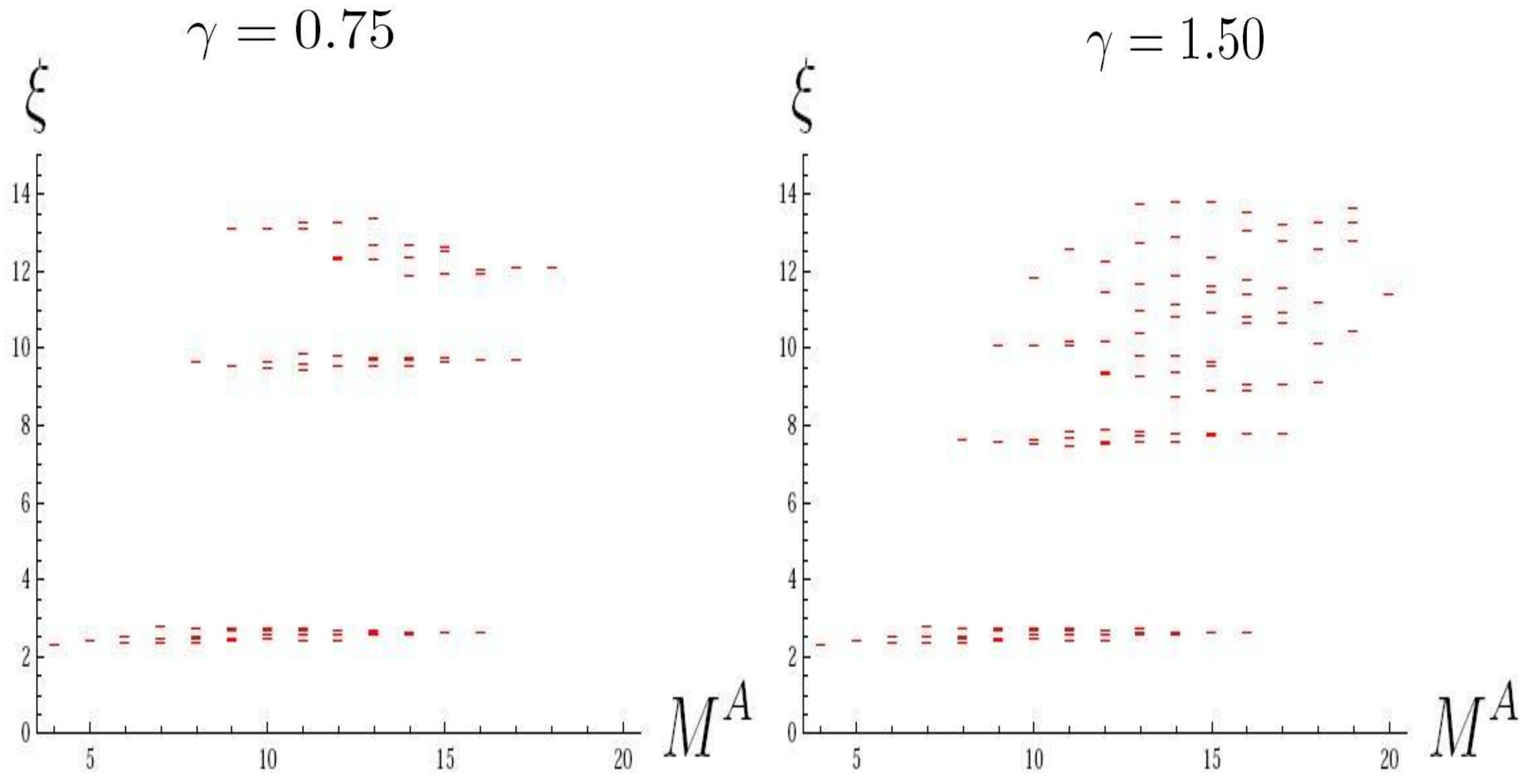
The ES of Haldane-Shatry purely consists of the universal low energy states as Laughlin on the QH sphere





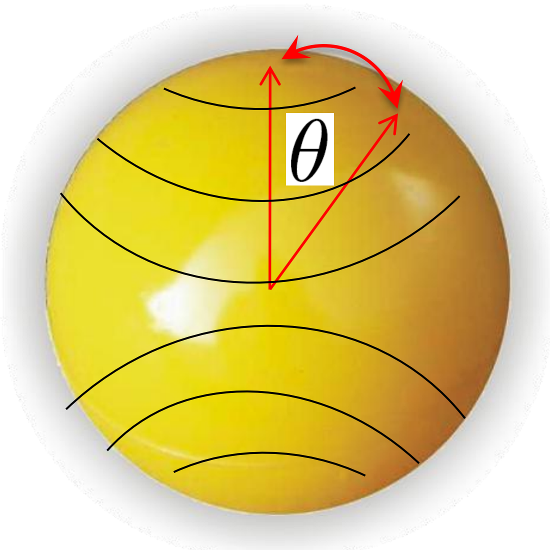
Logarithmic CFT corrections entering form the conform HS point

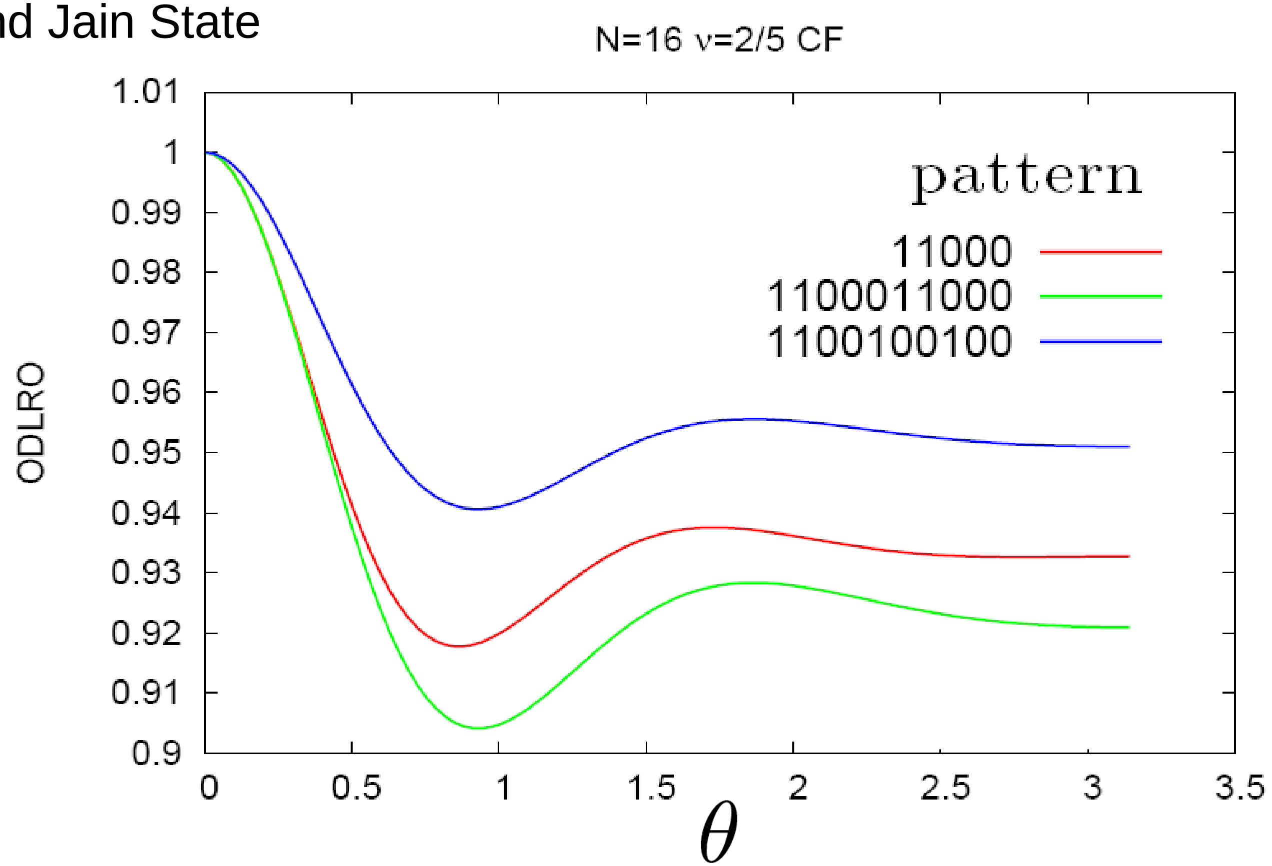
$$H_{\text{int}} = J_0 \gamma^2 \sum_{i<j} \frac{\vec{S}_i \cdot \vec{S}_j + \frac{1}{2}}{2 \sinh^2(\gamma |z_i - z_j|/2)}$$



Squeezed ODLRO can distinguish Gaffnian vs Jain/Coulomb

- States at 2/5: Gaffnian and Jain State





$|\Psi\rangle$ is a given FQH wf

A_{pattern} kills a given pattern

$|\psi_{\theta=0}\rangle = A_{\text{pattern}}|\Psi\rangle$

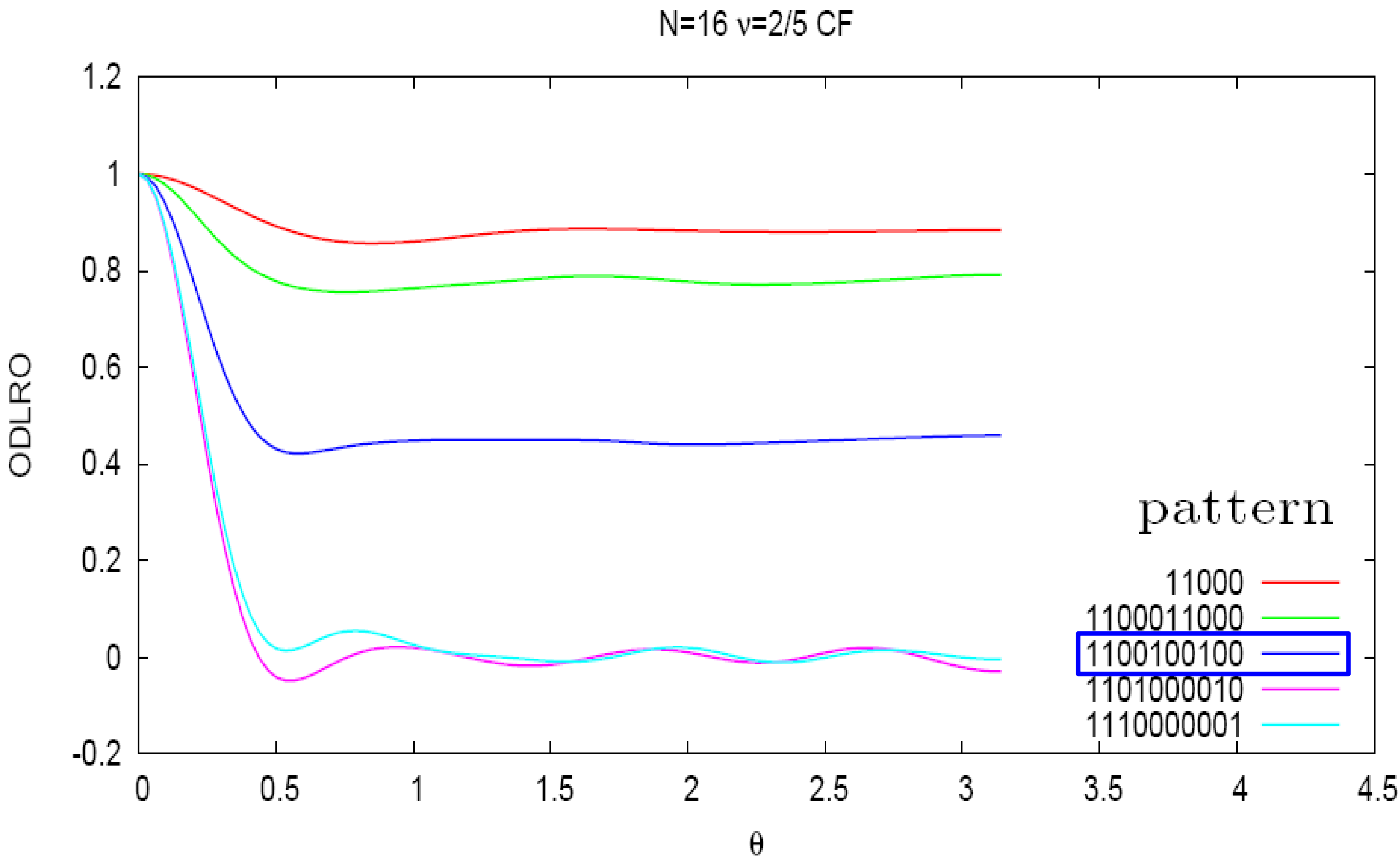
$|\psi_{\theta}\rangle = R(\theta)|\psi_{\theta=0}\rangle$

$R(\theta)$ rotation matrix

1100011000110001100011… = Gaffnian

1100100101001010010100101001 1 = Jain

Squeezed ODLRO can distinguish Gaffnian vs Jain/Coulomb



11000110001100011000 11… = Gaffnian

1100100101001010010100101 0010011 = Jain

Conclusions

- Model states without exact diagonalization lead to 1000 increase in speed-time

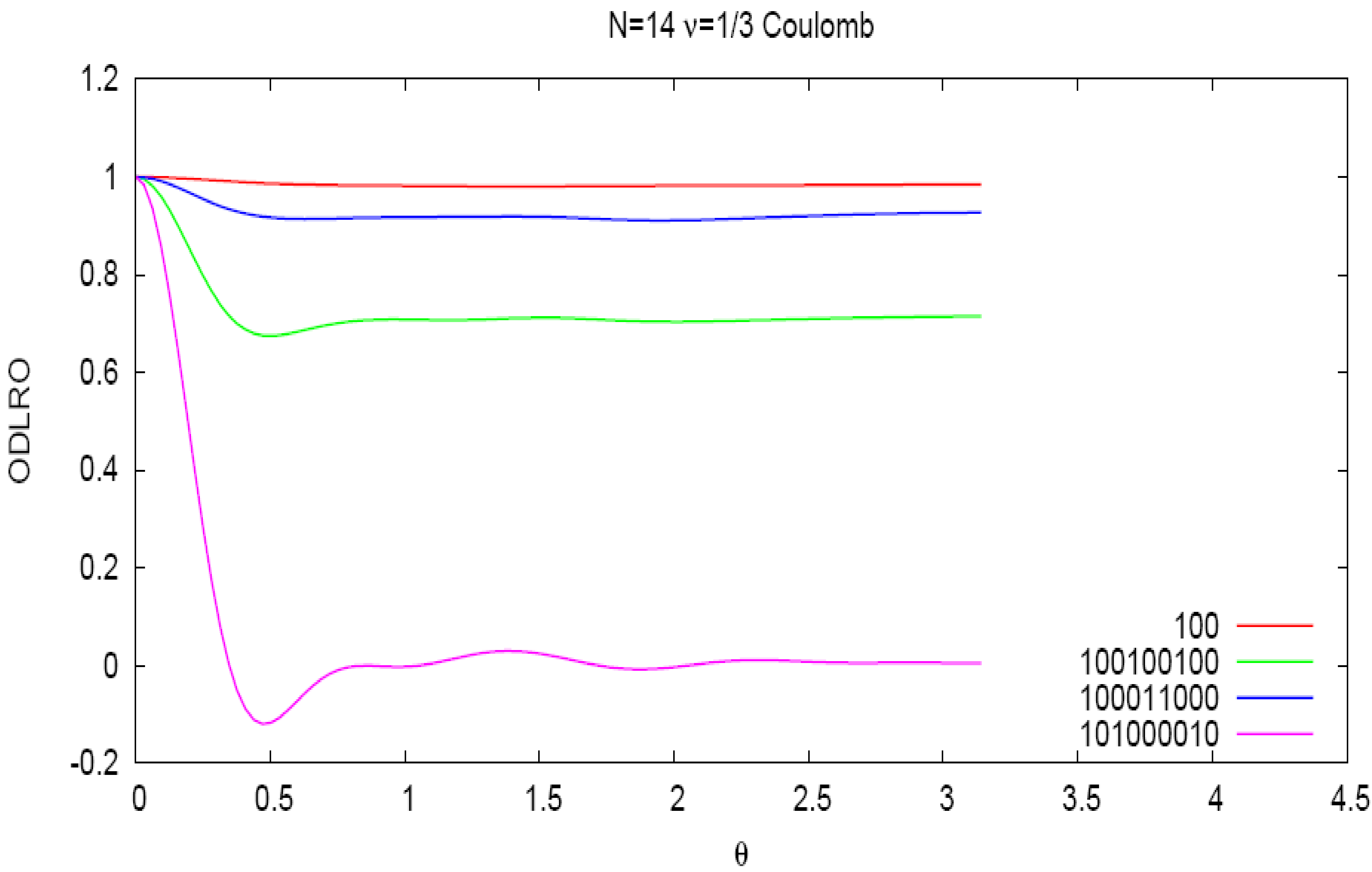
- New product rule for weights of the model states

- Squeezed ODLRO for the generic FQH states can determine the pattern of zeroes

- New muscle shows difference between Jain and Gaffnian

- Entanglement spectrum for large sizes also shows difference (see N Regnault's talk)

- We can now see that the nonunitary Gaffnian fails to screen



N=14 ν=1/3 Coulomb
ODLRO
θ
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
100
100100100
100011000
101000010

Perfect ODLRO is special case of Product Rule

ODLRO statement (for Laughlin states) $|N\rangle_L^m$: $\psi(z)|N+1\rangle_L^m = (\prod_{i=1}^{N}(z-z_i))^m |N\rangle_L^m$

- Killing an electron in the N+1 particle Laugh = creating m fluxes in the N part Laugh
- This is just the product rule on m orbitals:

$|N\rangle_L^m$

1001001001…1001001

0110001001…1001001

0110000110…0011100

0101001010…0011100

…

Laughlin state is a perfect BEC of the pattern 100

(Read, Girvin Macdonald, Haldane Rezayi, 1990's)

- This generalizes ODLRO to non-abelian states (For N particle MR, kill 2 electrons close to the north pole, you get 4 fluxes on top of the N-2 particle MR): Moore-Read state is perfect condensate of pattern 1100

There's Much More Than ODLRO

Can apply Read ODLRO twice (for Laughlin states) from N to N-2 particle states:

- **ASSUMPTION:** generic states also have good Squeezed ODLRO.

$|N - 1\rangle_L^m$

1001001…1001001

1001001…0011100

…+ many other slaters

- FQH states are BECs of squeezed configuration numbers

- True meaning of ODLRO: measures clustering conditions of FQH states (BAB and Haldane, 2007, Wen and Wang 2008)!

100100 = twice the ODLRO

011000 = squeezed ODLRO (new). By product rule, removing these configurations from the N+1 particle state also gives the EXACT N-1 particle state

- Laughlin states have ODLRO for
  100100100
  011000100 BUT NOT 101000010
  100011000 FOR 110000001
  010101000
  001110000