THEORETICAL MODELING OF THE RWM FEEDBACK CONTROL CONSIDERING NEOCLASSICAL TOROIDAL VISCOSITY AND ERROR FIELD PENETRATION

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1. Overview

Two milestones have been proposed for the present year to be studied and solved:

11. Investigation of the RWM stability considering neoclassical viscous torque influence and following the penetration and amplification of non-resonant error field driven modes
12. Determination of the plasma toroidal angular equation of motion

Within the frame of the first objective, a model for the influence of the error field penetration on the stability of the resistive wall modes (RWM) that contains non-resonant error field effects has been developed. The aim of this objective was to demonstrate in an explicit manner the destabilizing influence of the magnetic error fields on the RWM and its corresponding neighboring modes (i.e. error field penetration) at marginal stability for a large aspect-ratio, low beta, weakly shaped tokamak plasma. The pattern of the error field penetration lies on the joint effect of the corresponding electromagnetic torque and the viscous torques (due to neoclassical toroidal viscosity-NTV) at the edge and within the bulk of plasma at the level of the so called resonant rational surfaces (the wave vector of the magnetic perturbation error is perpendicular to the equilibrium magnetic field at rational surfaces). Whereas the single mode theory explains the appearance of localized magnetic torques and, consequently, the localized braking of the plasma rotation via NTV, the observed global braking of the plasma rotation requires a multimode analytic model to be developed. The mode coupling effect appear to have a major role in explaining the non-resonant (coupled) error fields penetration, the global braking of plasma rotation and consequently, the RWMs destabilization near marginal stability.

The second objective goal we have fulfilled was to clearly illustrate the deceleration and finally the toroidal braking of the tokamak plasma toroidal rotation due to non-resonant magnetic error field strength. As a continuation of the first objective, we have obtained a plasma coupled analytic toroidal angular equation of motion at the level of each rational surface inside the plasma with respect to the coupling coefficients between neighboring modes.

In the first objective's theoretical model, starting from the MHD instabilities dispersion relation, the solid increase of the harmonic perturbed flux amplitudes of the modes as the marginal stability is approached has been demonstrated. The error field presence is responsible for the above phenomenon. However the growth rate description type of the instabilities behavior (dispersion relation) is unable to describe the intrinsic influence of the magnetic error field on the plasma neoclassical toroidal viscosity (NTV). The error field augments the NTV destabilizing influence on MHD instabilities. In other words, the electromagnetic torques that develop at the levels of inner plasma inertial layers (at rational surfaces), due to the error field coupling phenomena, increase the destabilizing effect of the NTV torques at the non-ideal MHD layers we have mentioned. Consequently, the non-resonant (i.e. coupled) error field increases the NTV influence that brakes the toroidal plasma rotation locally, at the level of every plasma inner rational surface. To describe the above processes and explicitly illustrate the global braking of the toroidal plasma rotation a dynamic theoretical model has been built.

2. Detailed results

2.1. Investigation of the RWM stability considering neoclassical viscous torque influence and following the penetration and amplification of non-resonant error field driven modes

A multimode analytic cylindrical model has been developed starting from plasma equations that includes neoclassical, sound wave and dissipative particle collisions charge exchange effects. The combined effect of dissipative stabilization and neoclassical viscous torque destabilization is due to the toroidal component of the perturbed parallel neoclassical viscosity (NTV), $-\mathbf{z}\cdot\left(\nabla\cdot\mathbf{\Pi}_{\|}\right)$, in a (r,θ,z) cylindrical coordinates system. The parallel stress tensor is $\mathbf{\Pi}_{\|}=-3\eta_0\left(\mathbf{nn}-\vec{\mathbf{I}}/3\right):\nabla\mathbf{v}\left(\mathbf{nn}-\vec{\mathbf{I}}/3\right)$, where η_0 is the parallel ion viscosity, $\mathbf{n}=\mathbf{B}/B$ and $\mathbf{B}$ and $\mathbf{v}$ are the equilibrium magnetic field and the perturbed plasma velocity, respectively.
Keeping into account the expression of the safety factor $q(r)\cong rB_z/R_0B_\theta$, for a low inverse aspect ratio $\varepsilon=r/R_0$, the normalized poloidal component of the equilibrium magnetic field is $B_\theta/B_{z0}=\varepsilon/[q(1+\varepsilon\cos\theta)]+\vartheta(\varepsilon)$. As a result, the total normalized equilibrium magnetic field is

$$\frac{\mathbf{B}}{B_{z0}}=\frac{1}{1+\varepsilon\cos\theta}\left(0,\frac{\varepsilon}{q},1\right)+\vartheta(\varepsilon) \tag{1}$$

where $B_{z0}=B(0,\theta,z)$. The θ-dependence of $\mathbf{B}$ provides a small deviation from the cylindrical geometry allowing the inclusion of neoclassical effects into the calculus.
The linearized MHD equations we use are

$$\begin{aligned}&\gamma\mathbf{b}=\nabla\times\left(\mathbf{v}\times\mathbf{B}\right)\\&(\gamma+\upsilon)\rho\mathbf{v}=-\nabla p-\mathbf{z}\left(\mathbf{z}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\right)+\mathbf{j}\times\mathbf{B}+\mathbf{J}\times\mathbf{b}\\&\gamma\,p=-\rho c_s^2\nabla\cdot\mathbf{v}\end{aligned} \tag{2}$$

$\mathbf{b},p$ and $\mathbf{j}$ are the perturbations of the magnetic field, pressure and current density, respectively. ρ and c_s are the plasma mass density and the speed of sound in plasma. υ parameterizes the energetic exchanges in plasma as a result of the collisions of the particles. Suppose that all the perturbations have an $\exp(\gamma\,t)=\exp\left[\gamma_0 t+i\left(m\Omega_\theta-n\Omega_z\right)\right]$ dependence, where γ_0 is the instability growth rate, $\Omega_{\theta,z}$ the poloidal and toroidal rotation speeds at the level of (m,n) rational surface. We use the following parametrization of the plasma velocity:

$$\mathbf{v}=\gamma\nabla\Phi\times\mathbf{n}+v_{\|}\mathbf{n} \tag{3}$$

where Φ is the poloidal perturbed magnetic flux and $v_{\|}$ the parallel component of the plasma velocity. A Fourier-type description of all the perturbations is used ($\phi=z/R_0$)

$$x=\sum_{m,n}x^{mn}\exp\left[i\left(m\theta-n\phi\right)\right] \tag{4}$$

The feedback "component" of the system of equations has been subtracted from [1] to complete the global system of linearized equations, whereas the plasma part of the problem has been replaced by a new model, considering neoclassical effects. The feedback system for stabilizing the RWMs consists of a passive and an active shell. The assumption of a thin resistive shell approximation is taken into account. The passive shell consists of two different incomplete and nonoverlapping resistive shells (aluminum and stainless steel) disposed in an alternative manner and angular toroidally uniform. The active system consists of a number of rectangular, radial thin coils and detectors centered at the same local coordinates, the magnetic flux measured by the detector being amplified and fed back into the coils.

A complete linearized system that governs the RWM stability has been obtained:

$$\begin{aligned} &\sum_{j=m_1}^{m_2}\sum_{k=n_1}^{n_2}\sum_{i=0}^{5}\gamma_0^i\left(P_{jki}^{mn}\Psi_a^{jk}+Q_{jki}^{mn}\Psi_a^{jk'}\right)=0 \\ &\sum_{j=m_1}^{m_2}\sum_{k=n_1}^{n_2}\sum_{i=0}^{2}\gamma_0^i\left(F_{jki}^{mn}\Psi_a^{jk}+G_{jki}^{mn}\Psi_a^{jk'}\right)=0 \end{aligned} \tag{5}$$

P_{jki}^{mn} and Q_{jki}^{mn} contain all the information about plasma influence, whereas F_{jki}^{mn} and G_{jki}^{mn} describe the feedback part of the problem. Ψ_a^{jk} is the normalized perturbed flux function at the plasma boundary ($r=a$) and $\Psi_a^{jk'}\equiv d\Psi_a^{jk}/dr$. The poloidal number m spans the $[m_1,m_2]$ interval and the toroidal number n the $[n_1,n_2]$ interval taken into account. From here, applying the zero determinant condition and using the Leibniz description of the determinants, the following polynomial equation in RWM growth rate has been obtained, in the absence of magnetic error fields:

$$\sum_{i=1}^{8L+1}\gamma_0^{i+1}\sum_{\substack{l_1,\dots,l_{2L}=1\\ distinct}}^{2L}\mathrm{sgn}(l_1,\dots,l_{2L})\sum_{\substack{\alpha_1,\dots,\alpha_{2L}=1\\ \alpha_1+\dots+\alpha_{2L}=i+2L-1}}^{5}\prod_{s=1}^{2L}\Gamma_{s\alpha_s}^{l_s}=0 \tag{6}$$

where

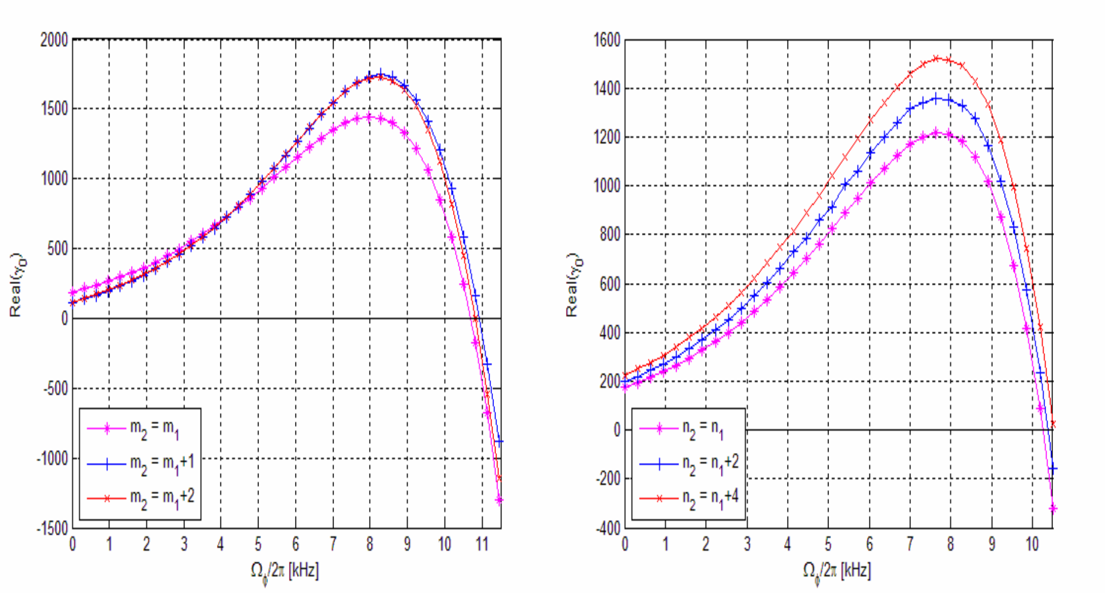
$$\begin{aligned} \Gamma_{s\alpha_s}^{l_s} = &\left[P_{s\alpha_s}^{l_s}\Theta(L-l_s)+Q_{s\alpha_s}^{l_s}\Theta(l_s-L+1)\right]\Theta(L-s)\Theta(s-1)+ \\ &\left[F_{s\alpha_s}^{l_s}\Theta(L-l_s)+G_{s\alpha_s}^{l_s}\Theta(l_s-L+1)\right]\Theta(2L-s)\Theta(s-L-1) \end{aligned} \tag{7}$$

Θ is Heaviside unit step function and $L=(m_2-m_1+1)(n_2-n_1+1)$. A new index ordering has been adopted so that the $(m+\alpha,n+\beta)$ mode becomes the $[(m_2-m_1+1)\beta+\alpha+1]$-th mode in the new ordering. The relations between the old and the new index are given by $l_s=j-m_1+1+(k-n_1)(m_2-m_1+1)$ and $s=m-m_1+1+(n-n_1)(m_2-m_1+1)$ [1].

The RWM stability is not affected by the small amplitude error fields influence as long as the RWM is far from marginal stability. The following reasoning has been adopted: the main unstable harmonic (RWM) equation has been obtained taking into account an unlimited number of poloidal neighboring modes. We follow the development of perturbed MHD equations (2) around the ε factor, considered small in the low aspect ratio approximation. The coupling is provided by the neoclassical dependence of the equilibrium magnetic field (1). In comparison with the main unstable harmonic, the adjacent modes are of $\vartheta(\varepsilon)$ order less unstable and the instability decreases as the modes are more distant from the central harmonic. This is the physical reason for treating, on the other hand, the adjacent modes (less unstable) as single modes in separate equations. Nevertheless, the feedback system that surrounds the plasma couples the perturbed modes both poloidally and toroidally.

Within the frame of the first part of the phenomena, the role of the NTV is double: destabilizes the RWM by neighboring mode coupling, but also stabilizes the modes by dissipation due to neoclassical flow damping. The profound global destabilizing effect of the NTV by braking plasma rotation as the RWM attains its marginal stability will be showed in the next report.

Figure 1a presents the dependence of the $(3,1)$ RWM growth rate with the edge plasma rotation under the effect of sideband poloidal mode coupling for the HBT-EP tokamak case. Figure 1b presents the same dependence for sideband toroidal mode coupling. The plasma parameters are found in [2], whereas the feedback parameters are those from [1]. The calculated relations (6) and (7) have been used.



(a) (b)

Figure 1. Growth rates of the (3, 1) unstable mode as a function of edge plasma normalized toroidal rotation in the absence of any sideband mode and in the presence of (a) one and two poloidal sideband modes and (b) two and four toroidal sideband modes.

The destabilizing effect of mode coupling and the stabilizing effect of plasma rotation are obvious.

By introducing static non-resonant error fields flux functions Ψ_{err}^{jk} in the absence of plasma, that correspond to (j,k) mode unstable at the level of its corresponding resonant rational surface, the following relation that includes field errors contribution at marginal stability has been obtained:

$$\sum_{j=m_1}^{m_2}\sum_{k=n_1}^{n_2}\sum_{p=0}^{2}\left[i\,\mathrm{Im}(\gamma_0)\right]^p\left(F_{jkp}^{mn}+G_{jkp}^{mn}\,z^{jk}\right)\Psi_a^{jk} =$$

$$= \sum_{j=m_1}^{m_2}\sum_{k=n_1}^{n_2}\frac{2}{k\hat{k}_0^{j}\left[az^{jk}-(1+1/\hat{k}_{10}^{j})\,r_{wa}\right]}\left[\sum_{p=1}^{2}\left[i\,\mathrm{Im}(\gamma_0)\right]^p\left(F_{jkp}^{mn}+G_{jkp}^{mn}\,z^{jk}\right)\right]\Psi_{err}^{jk} \qquad (8)$$

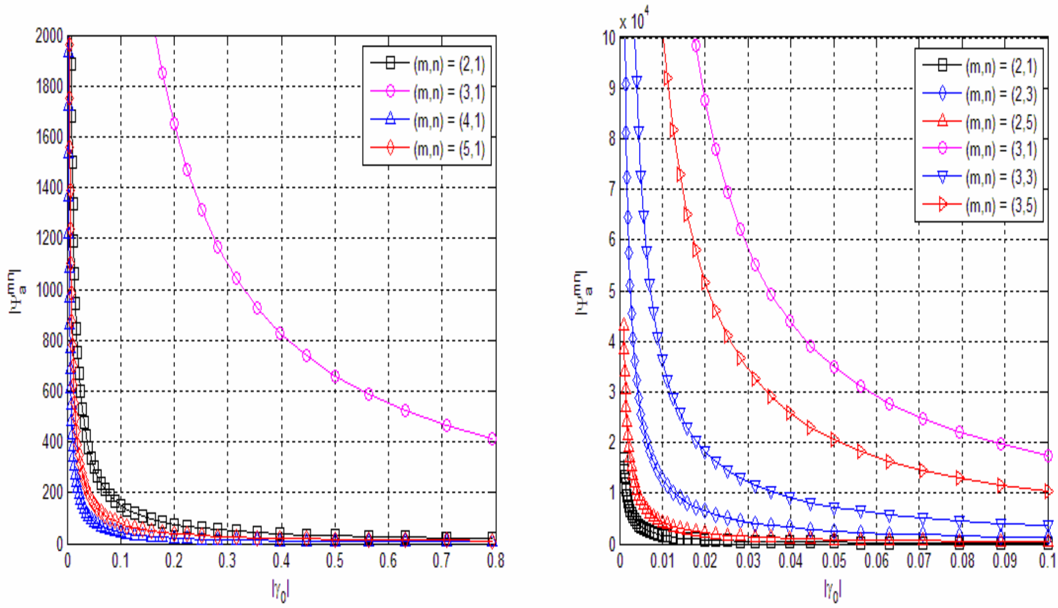
Ψ_a^{jk} and Ψ_{err}^{jk} are the renormalized perturbed plasma flux and the magnetic error flux at the boundary of the plasma respectively. $\hat{k}_{10}^{jk} = -\left[1-(r_w/a)^{-2j}\right]/2$, $z^{jk} = d\ln\Psi_a^{jk}/dr$ and $r_{wa} = \left[(r_w/a)^{j}-(r_w/a)^{-j}\right]^2/4$, where r_w is the radius of the surrounding resistive wall. $\mathrm{Im}(\gamma_0)$ is the imaginary part of γ_0, i.e. the real frequency of the mode and, according to the experimental results, is around one order of magnitude lower than the level of plasma rotation. At marginal stability, the approximation $\exp(\gamma_0 t_2)-\exp(\gamma_0 t_1)\cong\gamma_0(t_2-t_1)$ holds, $t_{1,2}$ being integration times for perturbed equations. In conclusion, for marginal stability approximation, time integrated perturbed equations are independent of integration time, for static error fields Ψ_{err}^{jk}.

By calculating the plasma level of rotation that corresponds to RWM marginal stability and its real frequency from RWM dispersion relation (6), the system of equations (5) becomes a non-homogeneous overdetermined system in order to obtain the z^{jk} unknowns. The system has a unique solution if zero characteristic determinants condition is applied. With $\mathrm{Im}(\gamma_0)$ and z^{jk}

obtained, using equation (8), the non-resonant error fields influence on RWM and its sideband modes can be pointed out.

It is known that mode locking together with global plasma rotation braking are the result of error fields influence predominantly localized near the outboard midplane of the plasma column. This localization facilitates the powerful coupling of poloidal perturbation modes. In the following, we shall assume sufficient plasma rotation levels as to overcome non-ideal behavior of the plasma in the vicinity of internal rational surfaces (driven magnetic islands).

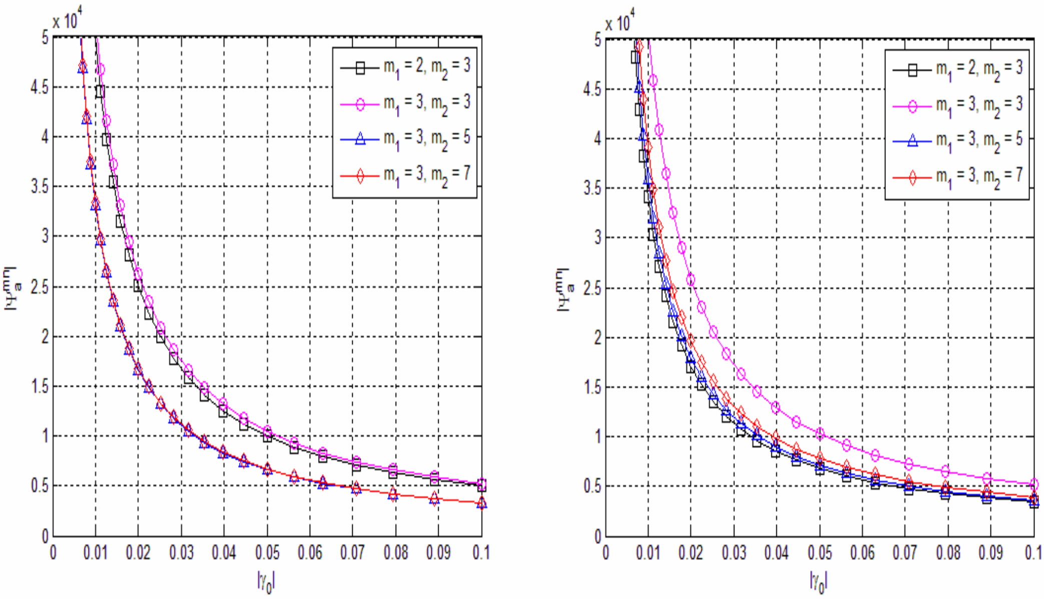
The non-resonant, global perturbed flux amplification is showed in Figures 2a and 2b. The central harmonic $(3,1)$ is accompanied by neighboring poloidal modes having $m = 2,..,5$ (Fig.2a). No toroidal coupling is considered. As $|\gamma_0|$ goes to zero (marginal stability), the perturbed flux calculated at plasma boundary $\left|\Psi_a^{mn}\right|$, associated to each mode taken into account grows asymptotically. For each corresponding error field magnetic flux Ψ_{err}^{jk}, it can be seen that the sideband modes whose poloidal numbers are more positive that the central resonant mode number grow slower. On the contrary, the modes with more negative poloidal mode number grow faster. However, the RWM (central harmonic) is the most unstable mode under the action of a non-resonant error field. Figure 2b shows the same above dependence for the case of several different number of toroidal sideband modes that accompany the RWM. Unlike the pure poloidal coupling case, the more positive toroidal sideband modes grow faster comparing with the more negative toroidal modes.



(a) (b)

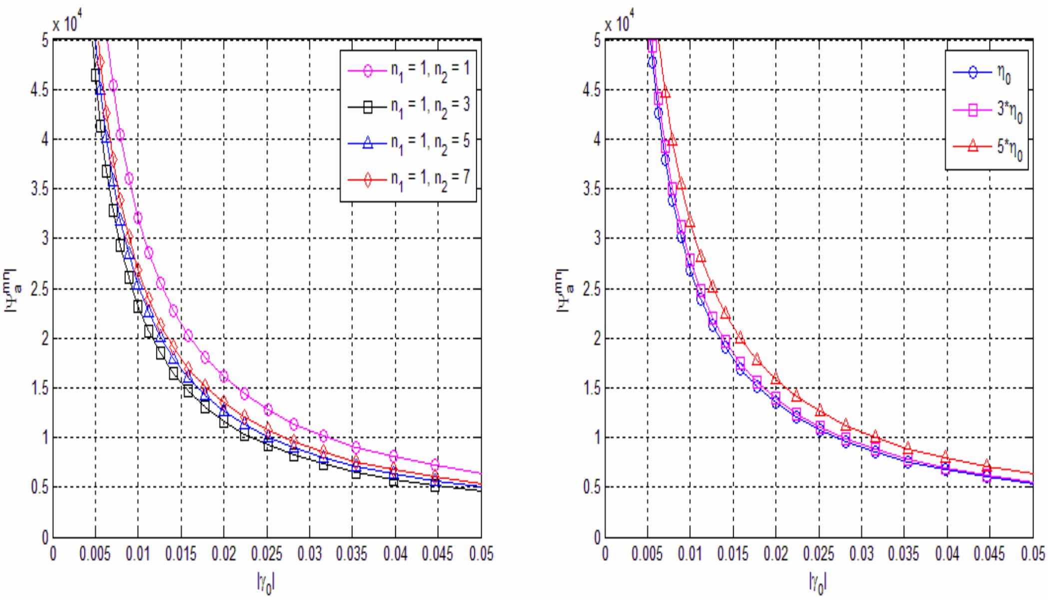
Figure 2. Central harmonic (3,1), (a) poloidal sideband perturbed flux amplitudes (m=2,..,5, no toroidal coupling) and (b) toroidal sideband perturbed flux amplitudes (m=2,3, n=1,..,5) as functions of $|\gamma_0|$ *at marginal stability, under the influence of a non-resonant error field.*

The poloidal coupling effect of neighboring modes on the central harmonic is shown in Figures 3a and 3b. To estimate the poloidal mode coupling influence on the central perturbed harmonic mode under the influence of the error fields, a different number of poloidal sideband modes has been considered. Figure 3a shows the central $\left|\Psi_a^{31}\right|$ dependence on its corresponding growth rate decreasing for different poloidal sideband modes taken into account. It can be pointed out that, in the absence of mode coupling, the resonant harmonic grows faster under the action of error field at marginal stability. Single mode theory is unable to determine the optimal error field spectrum to avoid mode locking. On the other hand, significant mode coupling with sideband modes that have more positive poloidal numbers is more effective than mode coupling with sideband modes having more negative poloidal mode numbers, in the absence of toroidal mode coupling.



(a) (b)

Figure 3. Central harmonic (3,1) perturbed flux amplitude as a function of $|\gamma_0|$ at marginal stability, under the action of a non-resonant error field in the absence and in the presence of one more negative, two and four more positive neighboring poloidal sideband modes without (a) and with (b) toroidal mode coupling $(n_1 = 1, n_2 = 3)$*, respectively.*



(a) (b)

Figure 4. (a) Central harmonic (3,1) perturbed flux amplitude as a function of $|\gamma_0|$ at marginal stability, under the action of a non-resonant error field in the absence and in the presence of two, four and six neighboring toroidal sideband modes, respectively (with poloidal coupling, $m_1 = 3, m_2 = 4$*); (b) Central harmonic (3,1) perturbed flux amplitude as a function of $|\gamma_0|$ at marginal stability for different values of NTV (*$m_1 = 3, m_2 = 4$, $n_1 = 1, n_2 = 7$*).*

If the toroidal mode coupling is also considered, the same above dependence changes significantly, as one can see in Figure 3b. Additional (toroidal) coupling involve higher

effectiveness for coupling with more negative modes rather than coupling with the more positive ones. Concerning this aspect, it can be pointed out that the strength of the mode coupling is an important factor in choosing the appropriate effective error field spectrum.
The toroidal coupling effect of neighboring modes on the central harmonic is shown in Figures 4a. The central harmonic (3,1) behavior at marginal stability is optimized, in the sense of less instability, by the coupling with the more positive toroidal neighboring modes, in the presence of an existing poloidal coupling.
However, an important aspect can be stated from the above cases: the central harmonic amplitude is lowered by any type and strength of mode coupling.
Figure 4b shows the expected NTV influence on modes destabilization and locking at marginal stability in the presence of non-resonant error fields: higher ion viscosity η_0 means higher amplitude for the central harmonic perturbed flux, due to the increase of the viscous torque and the corresponding joint effect with the perturbed electromagnetic torque. The same dependence is valid for every sideband mode, not only for the central harmonic, meaning the global braking influence of NTV at the level of every resonant rational surface.

Several conclusions can be tracked from the above dependencies. First, the optimal error field spectrum is dominated by the central resonant harmonic (Figure 2) and the sideband modes couple with the central harmonic mode in a manner that lowers the perturbed flux amplitude of the latter (Figures 3 and 4a). The sideband harmonics, as marginal stability is approached, couple back with the central harmonic to reduce its amplitude. Second, under significant mode coupling conditions, the optimal error field spectrum, in the sense of less instability, involves the prevalence of sideband neighboring modes that have more negative mode number comparing to the main harmonic (Figures 3b and 4a), under overall strong coupling conditions. Conversely, for the poloidal coupling case, in the absence of toroidal coupling (weak overall coupling), the more negative neighboring coupled modes are more ineffective than the corresponding more positive modes, in the sense of less instability (Figure 3a). With redefined notions of strong and weak coupling, one retrieves the conclusions found in the only paper, recently published [3], that investigate analytically, in a different manner, the non-resonant (multimode) error field influence on the RWM stability (only the poloidal case is investigated). The present model is able to extend the investigation to the toroidal case. Due to the calculus that includes both poloidal and toroidal mode coupling, the optimal error field spectrum can be determined. Finally, although NTV has a stabilizing dissipative effect as long as RWM in far from marginal stability (where small error fields have practically no effect on RWM stabilization), its global destabilizing aspect under error fields action as marginal stability is approached has been proved (Figure 4b).

References

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[2] R. Fitzpatrick and J. Bialek, *Phys. Plasmas* 13 (2006) 072512.
[3] R. Fitzpatrick, *Phys. Plasmas* 16 (2009) 032502.

2.2. Determination of the plasma toroidal angular equation of motion

From the equations that describe the MHD instabilities behavior in vacuum [1], after a laborious calculus we have obtained the following dynamic system of differential equations:

$$\sum_{j,k}\left(\sum_{i=0}^{2} W_{mni}^{jk}\frac{\partial^i}{\partial t^i}\right)\left[u_{ps}^{jk}\frac{\partial \Psi^{jk}}{\partial r}+u_{tt}^{jk}\frac{\partial^2\Psi^{jk}}{\partial t^2}+\left(\sum_{i=0}^{1}u_{ti}^{jk}\Omega_{z0}^{i}\right)\frac{\partial\Psi^{jk}}{\partial t}+\left(\sum_{i=0}^{2}u_{si}^{jk}\Omega_{z0}^{i}\right)\Psi^{jk}\right]$$
$$=\sum_{j,k}\left(\sum_{i=0}^{1}A_{mni}^{jk}\frac{\partial^i}{\partial t^i}\right)\Psi^{jk}+\sum_{j,k}E_{mn}^{jk}\Psi_{error}^{jk} \tag{9}$$

The above equations show the interaction of the plasma MHD instabilities with the plasma external structures consisting of a feedback system (a passive and an active shell). Within the assumption of a thin shell approximation, the passive shell consists of two different incomplete and non-overlapping resistive shells (aluminum and stainless steel) disposed in an alternative manner and resistive toroidally uniform. The active system consists of a number of rectangular, radial thin coils and detectors centered at the same local coordinates, the magnetic flux measured by the detector being amplified and fed back into the coils. Ψ^{jk} is the perturbed magnetic flux that corresponds to the (j,k) mode perturbation. The parameters W_{mni}^{jk} , u_{ps}^{jk}, u_{tt}^{jk}, u_{ti}^{jk} , u_{si}^{jk}, A_{mni}^{jk} and E_{mn}^{jk} give all the information about the position, disposal, resistive inhomogeneity and amplification amplitude concerning the feedback system that surrounds the plasma column. Ω_{z0} is the plasma toroidal rotation in the MHD regions of the plasma and Ψ_{error}^{jk} describes the magnetic error field spectrum.

From the plasma equations that describe the MHD instabilities behavior we have derived the following dynamic system of differential equations:

$$\sum_{j,k}\Big[s_{jk4}^{mn}\frac{\partial^4}{\partial t^4}-\left(4ik\Omega_{z0}s_{jk4}^{mn}-s_{jk3}^{mn}\right)\frac{\partial^3}{\partial t^3}-\left(6k^2\Omega_{z0}^2 s_{jk4}^{mn}+3ik\Omega_{z0}s_{jk3}^{mn}-s_{jk2}^{mn}\right)\frac{\partial^3}{\partial t^3}+$$
$$+\left(4ik^3\Omega_{z0}^3 s_{jk4}^{mn}-3k^2\Omega_{z0}^2 s_{jk3}^{mn}-2ik\Omega_{z0}s_{jk2}^{mn}+s_{jk1}^{mn}\right)\frac{\partial}{\partial t}+k^4\Omega_{z0}^4 s_{jk4}^{mn}+ik^3\Omega_{z0}^3 s_{jk3}^{mn}- \tag{10}$$
$$-k^2\Omega_{z0}^2 s_{jk2}^{mn}-ik\Omega_{z0}s_{jk1}^{mn}+s_{jk0}^{mn}\Big]\Psi^{jk}+\sum_{j,k}\left\{s_{jki}^{mn}\rightarrow t_{jki}^{mn},\ i=0,..,4\right\}\frac{\partial\Psi^{jk}}{\partial r}=0$$

s_{jki}^{mn} and t_{jki}^{mn} are parameters that describe the intrinsic behavior of the plasma, including the viscous stress tensor influence.

To solve the complete system consisting of the above two systems of equations in Ψ^{jk} and $\partial\Psi^{jk}/\partial r$ the jump of the radial derivative of the magnetic flux perturbation over the inertial layer is needed. Chang et al. [2] derived an analytic formula valid in cylindrical geometry. Adapted for the equilibrium magnetic field structure used here,

$$\frac{\mathbf{B}}{B_{z0}}=\frac{1}{1+\varepsilon\cos\theta}\left(0,\frac{\varepsilon}{q},1\right)+\vartheta(\varepsilon)$$

(where $B_{z0}=B(0,\theta,z)$, q is the safety factor and θ the poloidal angle) the $\partial\Psi^{jk}/\partial r$ jump becomes

$$\Delta\Psi_s^{mn}=\frac{\partial\Psi_{s+}^{mn}}{\partial r}-\frac{\partial\Psi_{s-}^{mn}}{\partial r}=\frac{m'\pi}{r_s\tan(\pi\sigma_s/2)}\left(\sigma_s\delta_{m'm}+\sum_{m'}\alpha_{m'm}\right)\Psi_s^{m'n} \tag{11}$$

The calculated parameters σ_s and $\alpha_{m'm}$ are

$$\sigma_s=-\frac{q_s}{mq'\sqrt{1+\varepsilon^2/q_s^2}}\left[\frac{q_s'}{q_s}\left(3+\frac{r_s q_s''}{q_s'}-\frac{2r_s q_s'}{q_s}\right)+\frac{\varepsilon}{R_0 q_s\left(1+\varepsilon^2/q_s^2\right)}\left(2-\frac{r_s q_s'}{q_s}\right)\left(1-\frac{r_s q_s'}{q_s}\right)\right]$$

$$\alpha_{m'm} = \frac{q_s\left(2 - r_s q_s'/q_s\right)}{m' q_s' R_0 \sqrt{1+\varepsilon^2/q_s^2}} \int_0^{2\pi} \frac{\cos\theta \, e^{-i(m'-m)\theta}}{1+\varepsilon\cos\theta} d\theta \tag{12}$$

r_s is the minor radius that corresponds to the rational surface where the inertial layer develops, $\varepsilon = r_s / R_0$ and R_0 is the major radius of the plasma. $\alpha_{m'm}$, i.e. the mode coupling parameter, is the main parameter responsible for error field penetration and NTV torque augmentation.
Using all the relations above, we are able to obtain the following Laplace transformed complete system of equations:

$$\begin{aligned} &\sum_{j,k}\left[\left(\sum_{i-0}^{4}\tau^i p_{mni}^{jk}\right)\overline{\Psi_s^{jk}} + \left(\sum_{i-0}^{4}\tau^i q_{mni}^{jk}\right)\overline{\Psi_{s+}^{jk}{}'}\right] = 0 \\ &\sum_{j,k}\left[\left(\sum_{i-0}^{4}\tau^i f_{mni}^{jk}\right)\overline{\Psi_s^{jk}} + \left(\sum_{i-0}^{2}\tau^i g_{mni}^{jk}\right)\overline{\Psi_{s+}^{jk}{}'}\right] = \sum_{j,k}\frac{1}{\tau}E_{mn}^{jk}\Psi_{error}^{jk} \end{aligned} \tag{13}$$

where $\overline{\Psi_s^{jk}} = L\left(\Psi_s^{jk}(r,t)\right)$, p_{mni}^{jk}, q_{mni}^{jk} are plasma explicit parameters and f_{mni}^{jk} , g_{mni}^{jk} feedback system explicit parameters. Hereinafter we use the Leibniz description of the determinants we have developed in [1] for the above system of equations

$$\Delta_s \equiv \sum_{k=1}^{8L+1}\tau^{k-1} \sum_{\substack{l_1,\ldots,l_{2L}=1 \\ distinct}}^{2L} \mathrm{sgn}\left(l_1,\ldots,l_{2L}\right) \sum_{\substack{i_1,\ldots,i_{2L}=1 \\ i_1+\ldots+i_{2L}=k+2L-1}}^{5} \prod_{h=1}^{2L}\Gamma_{hi_h}^{l_h} \tag{14}$$

with

$$\begin{aligned} \Gamma_{hi_h}^{l_h} &= p_{hi_h}^{l_h} H(L-l_h)H(L-h) + q_{hi_h}^{l_h} H(l_h - L - 1)H(L-h) + \\ &+ f_{hi_h}^{l_h} H(L-l_h)H(h-L-1) + g_{hi_h}^{l_h} H(l_h - L - 1)H(h-L-1)H(2-i_h) \end{aligned} \tag{15}$$

H is the Heaviside unit step function and $L = \left(m_2 - m_1 + 1\right)\left(n_2 - n_1 + 1\right)$, where $m_1 \le m \le m_2$ and $n_1 \le n \le n_2$. Within the new index ordering the mode $(m+\alpha, n+\beta)$ becomes the $[(m_2 - m_1 + 1)\beta + \alpha + 1]$-th mode.
Their mutual relations are $l_h = j - m_1 + 1 + (k - n_1)(m_2 - m_1 + 1)$ and $h = m - m_1 + 1 + (n - n_1)(m_2 - m_1 + 1)$.
Conversely, we have

$j = l_h + m_1 - 1 - (m_2 - m_1 + 1)[(l_h - 1)/(m_2 - m_1 + 1)]$,

$m = h + m_1 - 1 - (m_2 - m_1 + 1)[(h - 1)/(m_2 - m_1 + 1)]$,

$j = l_h + m_1 - 1 - (m_2 - m_1 + 1)[(l_h - 1)/(m_2 - m_1 + 1)]$

$k = n_1 + [(l_h - 1)/(m_2 - m_1 + 1)]$ and $n = n_1 + [(h - 1)/(m_2 - m_1 + 1)]$

where [ ] denotes the integer part of a number. $\mathrm{sgn}\left(l_1,\ldots,l_{2L}\right)$ is the sign of the permutations.
The solutions of the Laplace transformed system of equations give the following Laplace transformed magnetic flux perturbations

$$\overline{\Psi_s^l} = \frac{\Delta_s^l}{\tau\Delta_s} \tag{16}$$

for $l = \overline{1,2L}$ (the index *s* refers to the rational surface that the inertial layer corresponds). Δ_s^l has the same expression as Δ_s with $\Gamma_{hi_h}^{l_h}$ replaced by the "error field term":

$$\Gamma_{hi_h}^{l_h} = p_{hi_h}^{l_h} H(L-l_h)H(L-h)\left(1-\delta_{l_h l}\right) + q_{hi_h}^{l_h} H(l_h - L - 1)H(L-h) +$$
$$+ f_{hi_h}^{l_h} H(L-l_h)H(h-L-1)\left(1-\delta_{l_h l}\right) + \left(\sum_{l_h=1}^{2L} E_h^{l_h} \Psi_{error}^{l_h}\right) H(L-l_h)H(h-L-1)\delta_{l_h l}\delta_{i_h 1} +$$
$$+ g_{hi_h}^{l_h} H(l_h - L - 1)H(h-L-1)H(2-i_h) \tag{17}$$

Using partial fraction decomposition and the inverse Laplace transform we are finally able to obtain the general expression of the *(m,n)* magnetic flux perturbation at rational surface r_s

$$\Psi_s^{jk}(r,t) = \sum_{j=1}^{L_0} \left[\frac{\left(\tau - \tau_j\right)\Delta_s^{jk}}{\tau \Delta_s} \right]_{\tau=\tau_j} \left(e^{\tau_j t} - 1\right) \tag{18}$$

where τ_j are the non-zero roots of the equation $\Delta_s = 0$ and L_0 is the corresponding polynomial degree. Recall that $l = j - m_1 + 1 + (k - n_1)(m_2 - m_1 + 1)$. The marginal stability has been chosen the initial condition, i.e. $\partial^i \Psi^{mn}(r_s, 0)/\partial t^i \cong 0$ for any $i \geq 0$. However, due to the low values of the error field, the solution did not essentially change if the following initial conditions are chosen: $\partial^i \Psi^{mn}(r_s, 0)/\partial t^i \cong 0$ for any $i \geq 1$ and $\Psi^{mn}(r_s, 0) \cong \Psi_{error}^{mn}(r_s)$.

Following the Braginskii [3] description of the viscous stress tensor into parallel (η_0), perpendicular (η_1 and η_2) and gyroviscous (η_3 and η_4) components, the tensor elements are:

$$\Pi_{ij} = -\eta_0 W_{0ij} - \eta_1 W_{1ij} - \eta_2 W_{2ij} + \eta_3 W_{3ij} + \eta_4 W_{4ij}$$

where

$$W_{0ij} = (3/2)\left(n_i n_j - \delta_{ij}/3\right)\left(n_k n_l - \delta_{kl}/3\right)W_{kl}$$
$$W_{1ij} = \left[\left(\delta_{ik} - n_i n_k\right)\left(\delta_{jl} - n_j n_l\right) + \left(\delta_{ij} - n_i n_j\right)n_k n_l / 2\right]W_{kl}$$
$$W_{2ij} = \left[\left(\delta_{ik} - n_i n_k\right)n_j n_l + \left(\delta_{jl} - n_j n_l\right)n_i n_k\right]W_{kl}$$
$$W_{3ij} = (1/2)\left[\left(\delta_{ik} - n_i n_k\right)\varepsilon_{jml} + \left(\delta_{jl} - n_j n_l\right)\varepsilon_{imk}\right]n_m W_{kl}$$
$$W_{4ij} = \left(n_i n_k \varepsilon_{jml} + n_j n_l \varepsilon_{imk}\right)n_m W_{kl}$$
$$W_{kl} = \partial v_k / \partial x_l + \partial v_l / \partial x_k - (2/3)\delta_{kl} \nabla \cdot \mathbf{v}.$$

$\mathbf{n} = \mathbf{B}/B$ where **B** is the equilibrium magnetic field and **v** is the fluid velocity. After the space integration of the inertial plasma layer toroidal equation of motion, within the cylindrical approximation and negligible poloidal rotation (compared to the the toroidal rotation), the only significant viscous coefficient that matters is the perpendicular coefficient η_2.

The toroidal angular equation of motion of the inertial layer is:

$$\rho \frac{\partial \Omega_z}{\partial t} = -\frac{\eta_2}{r_s^2 c_s^2} \Omega_z + \frac{1}{2\mu_0 r_s^2 R_0 c_s} \sum_{m,n} n \,\mathrm{Im}\left(\Delta \Psi_s^{mn} \Psi_s^{mn*}\right) \tag{19}$$

ρ is the layer mass density, Im is the imaginary part and * denotes the complex conjugate of a number. $c_s = \delta_s / r_s$ where δ_s is the layer width. δ_s has been chosen as a fixed parameter, although magnetic islands width theory exhibits an explicit perturbed magnetic flux dependence on the former. The mentioned dependence is not relevant for our purpose and is beyond the scope of the present work. The first term on the left-hand side of the equation corresponds to the NTV torque that acts on the inertial layer whereas the second term defines the error field electromagnetic torque that finally increases the destabilizing influence of the NTV.

Finally, after a straightforward calculus we have obtained the following analytic solution of the above equation

$$\Omega_z(t) = \Omega_{z0}\exp\left(-\frac{\eta_2}{\rho r_s^2 c_s^2}t\right) - \frac{\pi c_s}{2\mu_0 R_0^2}\sum_{\substack{m,m',n \\ m\neq m'}}\frac{m'n}{\tan(\pi\sigma_s/2)}\times$$

$$\times\sum_{j,k=1}^{L_0}\mathrm{Im}\left\{\frac{\alpha_{m'm}\beta_j^{m'n}\beta_k^{mn*}}{\tau_j\tau_k^*}\left[\frac{\exp\left[\left(\tau_j+\tau_k^*\right)t\right]}{\eta_2+\rho\left(\tau_j+\tau_k^*\right)r_s^2c_s^2} - \frac{\exp(\tau_j t)}{\eta_2+\rho\tau_j r_s^2 c_s^2} - \frac{\exp(\tau_k^* t)}{\eta_2+\rho\tau_k^* r_s^2 c_s^2} + 1\right]\right\} \tag{20}$$

where $\beta_j^{mn} = \lim_{\tau\to\tau_j}\left[(\tau-\tau_j)\Delta_s^{mn}/\Delta_s\right]$. A few observations can be made. First, the presence of the magnetic error fields changes the NTV dependence of the toroidal plasma rotation. Second, all the spectrum of the non-resonant error field contribute to the toroidal deceleration of a certain plasma inertial layer but only due to the mode coupling process. Consequently, it seems that the dynamic theoretical model of the single mode theory is unable to explain the error field penetration and NTV non-resonant magnetic braking effects phenomena, starting from marginal stability.

To prove the above final relation correctness, for the well known Wesson profiles of plasma current and safety factor with $q_0 = 1.01$, $q_a = 2.95$ (the safety factors on the magnetic axis and plasma boundary, respectively), $m_1 = 1$, $m_2 = 3$, the dependencies below have been drawn. The figures clearly show the plasma deceleration and rotation braking under error field penetration and NTV destabilizing influence.

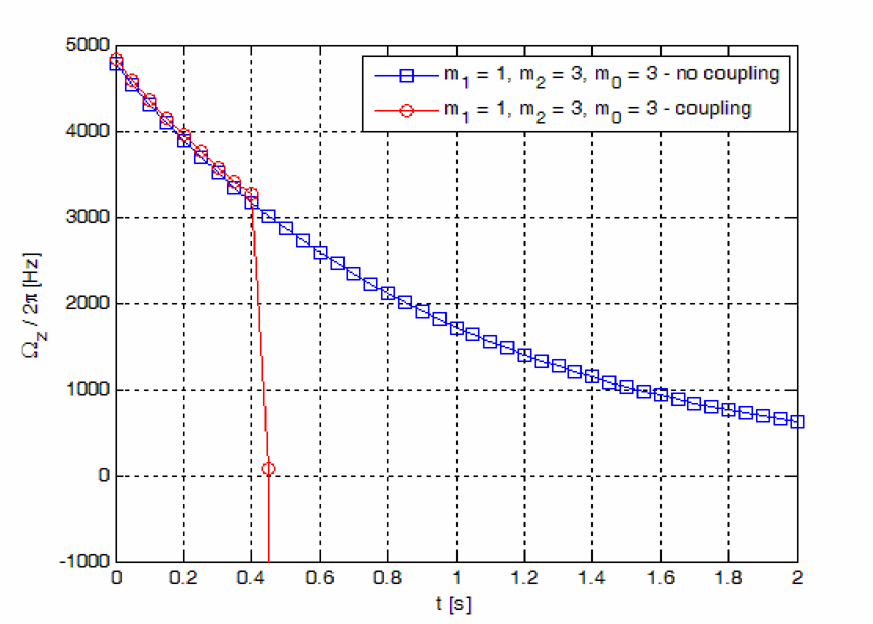

Figure 1: (3,1) inertial layer toroidal rotation rate (t is the normalized time, $q_s = m_0/n_0$)

Figure 1 presents the *(3,1)* inertial layer toroidal rotation rate (the plasma boundary layer that theoretically explains the appearance and behavior of the external kink modes) under error field action (round symbols) and in the absence of the error field mode coupling (square symbols). The abrupt toroidal rotation braking caused by the highly increased NTV influence under error field penetration process compared to the normal decreasing behavior of the rotation rate under weak feedback stabilizing conditions can be clearly observed.

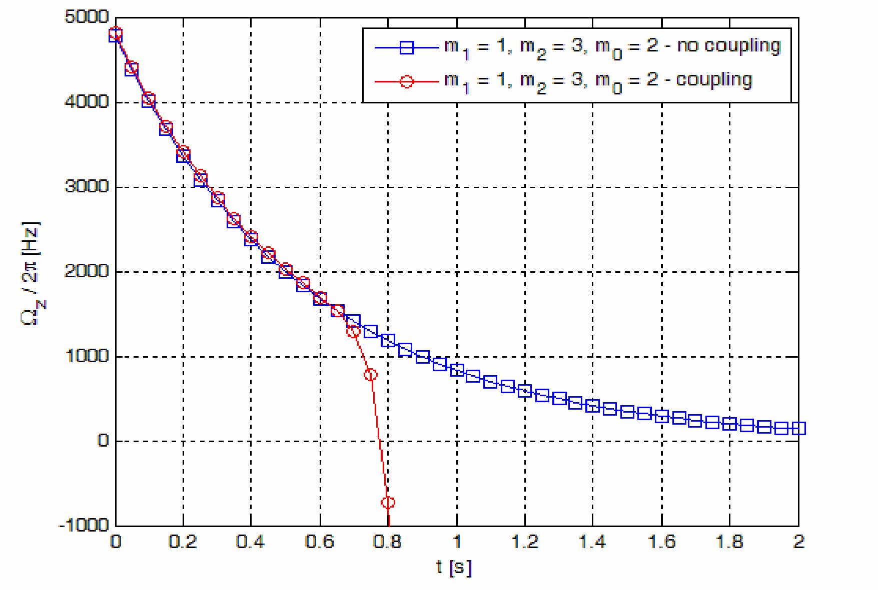

Figure 2: (2,1) inertial layer toroidal rotation rate (t is the normalized time, $q_s = m_0/n_0$)

Figure 2 shows the above dependency for the inner plasma *(2,1)* inertial layer. The same abrupt deceleration of the plasma layer under the error field influence and NTV amplification effect is showed.

To conclude, a full analytic dynamic description of the error field penetration and neoclassical toroidal viscosity non-resonant magnetic braking effects has been built. The analytic solutions for the components of the perturbed magnetic flux function have been found. The plasma toroidal angular equation of motion has been solved, an explicit analytic time-dependent solution being provided. We have showed that the error field penetration process is responsible for the increased destabilizing influence of the neoclassical toroidal viscosity on external and internal magnetohydrodynamic perturbations, caused by the global deceleration of the toroidal rotation of the plasma. The full spectrum of the non-resonant error field contributes to the damping of the rotation of every rational surface corresponding inertial layer due to the mode coupling phenomenon. The clear and explicit analytic obtained solution makes it possible to find the optimal less destabilizing error field spectrum as well as the optimal choice for the feedback parameters in order to provide stability.

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