Field theoretical methods in fluid and plasma theory

First part: Elements of mathematical structures

involved in the field theoretical formulation

of fluid and plasma models

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Main idea : there exist *preferred states* of the system.
The system makes transitions between these states.

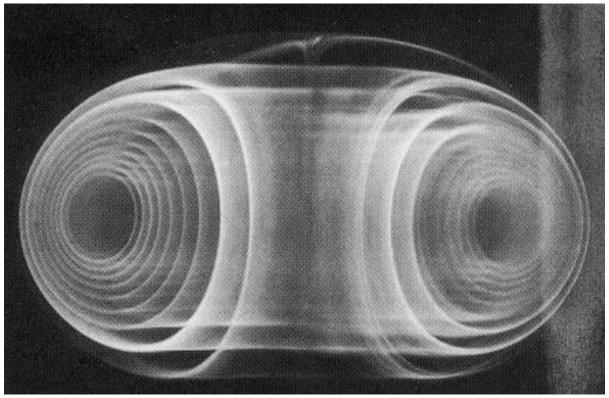
Quasi-coherent structures are observed in

- fluids (in oceans and in laboratory experiments)
- plasma (confined in strong magnetic field)
- planetary atmosphere ($2D$ quasi-geostrophic)
- non-neutral plasma (crystals of vortices)

There are common features suggesting to develop models based on the self-organization of the vorticity field. The fluids evolve at relaxation precisely to a subset of stationary states.

It is found that besides *conservation* there is also *action*

Coherent structures in fluids and plasmas (reality)

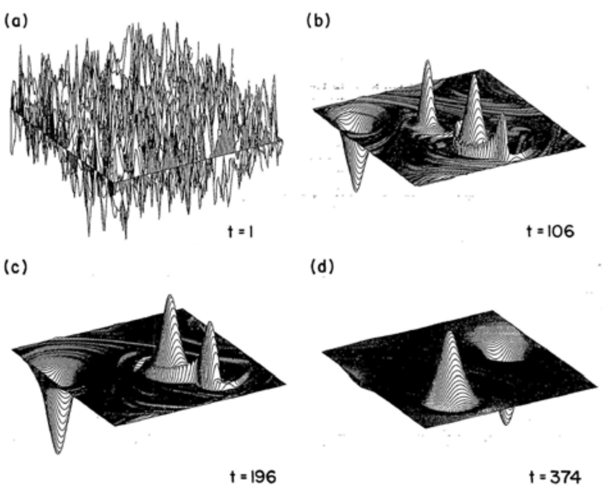


Rings of vorticity (Leonard 1998)

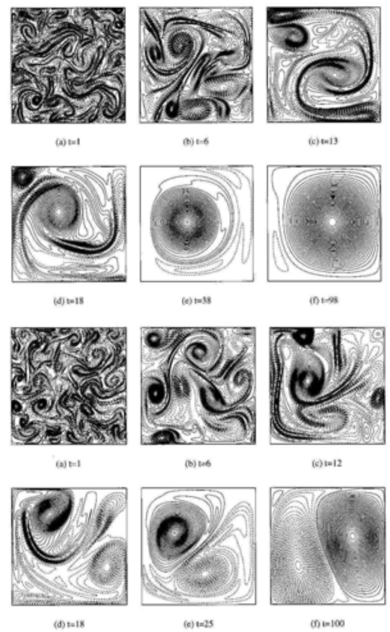
Nice tornado vortex.

Vortex ring emitted by the volcano Etna.

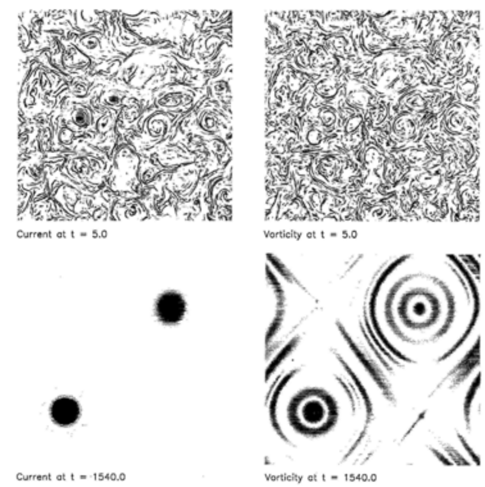
Coherent structures in fluids and plasmas (numerical)



Euler fluid: D. Montgomery et al. Phys. Fluids A4 (1992) 3.



Navier-Stokes fluid: H. Brands et al. Phys. Rev. E 60.



MHD : R. Kinney et al. Phys. Plasmas 2 (1995) 3623.

Compare the two approaches

Conservation eqs.

$$
\begin{aligned}
\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) &= 0 \\
mn\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} &= -\nabla p - \nabla\cdot\pi + \mathbf{F} \\
\frac{3}{2}n\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)T &= -\nabla\cdot\mathbf{q} - p\,(\nabla\cdot\mathbf{v}) - \pi : \nabla\mathbf{v} + Q
\end{aligned}
$$

Valid for : coffee, ocean, sun.

Lagrangian

$$
\begin{aligned}
\mathcal{L}\left(x^{\mu}, \phi^{\nu}, \partial_{\rho}\phi^{\nu}\right) &\rightarrow \mathcal{S} = \int dx\,dt\,\mathcal{L} \\
\frac{\partial}{\partial x^{\mu}}\frac{\delta\mathcal{L}}{\delta\left(\frac{\partial\phi^{\nu}}{\partial x^{\mu}}\right)} - \frac{\delta\mathcal{L}}{\delta\phi^{\nu}} &= 0
\end{aligned}
$$

Valid for : a single system.
Just give the initial state.

Lagrangians are preferable. But, how to find a Lagrangian ? See Phys.Rev.

Equivalence with discrete models

We will try to write Lagrangians *not* directly for fluids and plasmas but for equivalent discrete models.

An equivalent discrete model for the Euler equation

$$\frac{dr_k^i}{dt} = \varepsilon^{ij} \frac{\partial}{\partial r_k^j} \sum_{n=1, n\neq k}^{N} \omega_n G\left(\mathbf{r}_k - \mathbf{r}_n\right) \ , \ i,j = 1,2 \ , \ k = 1,N \tag{1}$$

the Green function of the Laplacian

$$G\left(\mathbf{r},\mathbf{r}'\right) \approx -\frac{1}{2\pi} \ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L}\right) \tag{2}$$

An equivalent discrete model for the CHM equation

The equations of motion for the vortex ω_k at (x_k, y_k) under the effect of the others are

$$\begin{aligned} -2\pi\omega_k \frac{dx_k}{dt} &= \frac{\partial W}{\partial y_k} \\ -2\pi\omega_k \frac{dy_k}{dt} &= -\frac{\partial W}{\partial x_k} \end{aligned}$$

where

$$W = \pi \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} \omega_i \omega_j K_0 \left(m \left| \mathbf{r}_i - \mathbf{r}_j \right| \right)$$

Physical model $\rightarrow$ point-like vortices $\rightarrow$ field theory.

This is a Lagrangian density

$$
\begin{aligned}
\mathcal{L} \quad = \quad & -\kappa\varepsilon^{\mu\nu\rho}\mathrm{Tr}\left(\partial_\mu A_\nu A_\rho - \frac{2}{3}A_\mu A_\nu A_\rho\right) \\
& -\mathrm{Tr}\left[\left(D^\mu\phi\right)^\dagger\left(D_\mu\phi\right)\right] \\
& -V\left(\phi^\dagger,\phi\right)
\end{aligned}
$$

- Free field dynamics, separate kinetic terms for *matter* and *gauge* fields
- Explicitely invariant to space-time symmetries
- Covariant derivatives (minimal coupling)
- nonlinear self-interaction

The potential $A^{\mu}\,(x,y,t)$ is a differential one-form called *connection*;

The field $F^{\mu\nu}$ is the differential two-form, called *curvature*

We need the concept of *fiber bundle* to introduce these definitions.

Differential manifolds

A differential manifold is a topological space, locally Euclidean, paracompact and connex.

The basic property of the manifold M is that it is locally Euclidean, which means that locally a manifold can be thought of as being a space like $\mathbf{R}^n$. We say that locally M is diffeomorphic to the space $\mathbf{R}^n$. It is defined a *chart*, which is the pair (U_x, ϕ) consisting of a neighborhood $U_x \in M$ of x and of a smooth function mapping this neighborhood on $\mathbf{R}^n$, $\phi : U_x \rightarrow \mathbf{R}^n$. Two charts that overlap define *transition functions*, $\phi_2 \circ \phi_1^{-1}$ form the open subset $\phi_1 (U_1 \cap U_2) \in \mathbf{R}^n$ onto the open subset $\phi_2 (U_1 \cap U_2) \in \mathbf{R}^n$. When these transition functions are differentiable the two charts are compatible.

The ensemble of charts is an *atlas*.

Tangent and cotangent spaces to a manifold

Few notions

- Point of a manifold, neighborhood, functions defined on the neighborhood and taking values in $\mathbf{R}^n$, the modul of real functions.
- Tangent vector in a point $x \in M$ (mapping from the modul of real functions to $\mathbf{R}^n$, or derivative of a real function *along* that vector), tangent space, vector field, tangent fiber space.
- linear mapping acting on tangent vectors. Differential forms, the cotangent space.

A vector is understood abstractly as the tangent vector to a curve at a point, the point and the curve being in the manifold.

The basis in the *tangent space* to a manifold is

$$\frac{\partial}{\partial x^\mu}$$

Let $T_p(M)$ be the tangent space in the point p at the manifold M. An arbitrary vector

$$V \in T_p(M)$$

can be written in the form

$$V = V^\mu \frac{\partial}{\partial x^\mu}$$

The basis of covectors (differential *one-forms*) in $T_p^*(M)$, the space cotangent to M, is formed by

$$\{dx^\mu\}$$

such that we have in the inner product defined on the spaces $T_p(M)$ and $T_p^*(M)$ the product

$$\left\langle dx^\mu, \frac{\partial}{\partial x^\nu} \right\rangle = \delta_\nu^\mu$$

Higher-order differential forms

The *wedge* product.

Combining the dx^{μ}'s antisymmetrically via the wedge product gives a convenient set of bases for the spaces of totally antisymmetric cotensor fields

$$\begin{aligned} dx^{\mu} \wedge dx^{\nu} &= dx^{\mu} \otimes dx^{\nu} - dx^{\nu} \otimes dx^{\mu} \\ dx^{\mu} \wedge dx^{\nu} \wedge dx^{\lambda} &= dx^{\mu} \otimes dx^{\nu} \otimes dx^{\lambda} \pm \text{permutations} \\ &\vdots \end{aligned}$$

called n - forms in n^{th} rank.

A p-form is a totally antisymmetric covariant tensor of rank p.

The space of all p-forms at the point x is

$$\Lambda^{p}(x)$$

This is a vector space. Its dimension is

$$\begin{aligned} \dim \Lambda^p &= \frac{n!}{p!\,(n-p)!} \\ &= \mathsf{C}_n^p \end{aligned}$$

It is equal with the number of combinations of n numbers taken in groups of p, without repetition.

The basis in the vector space $\Lambda^p(x)$ of p-forms is

$$\left\{dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}\right\} \text{ with } \mu_1 < \mu_2 < \cdots < \mu_p$$

The operator of exterior derivation.

The exterior derivative generally takes n- forms to $n+1$ forms is defined by

$$\frac{\partial}{\partial x^\mu} dx^\mu \wedge$$

and generates a minus sign when moved through forms of odd degree.

Example

$$d\alpha = \frac{\partial \alpha}{\partial x^{\mu}} dx^{\mu}$$

is the exterior derivative of a zero-form, *i.e.* a function, α is the differential of that function, expressed in the basis of independent differentials $(dx, dy, ...)$.

Example

$$d\left(\alpha_{\mu} dx^{\mu}\right) = \sum_{\nu} \frac{\partial \alpha_{\mu}}{\partial x^{\nu}} dx^{\nu} \wedge dx^{\mu}$$

The convention is that the new dx goes in front. More detailed, let us take a one-form α in a two-dimensional space

$$\alpha = P\left(x, y\right) dx + Q\left(x, y\right) dy$$

and calculate

$$\begin{aligned}
d\alpha &= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy\right)\wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy\right)\wedge dy \\
&= \frac{\partial P}{\partial y}dy\wedge dx + \frac{\partial Q}{\partial x}dx\wedge dy \\
&= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\wedge dy
\end{aligned}$$

The exterior product of a differential one-form with itself is zero

$$dx \wedge dx = 0$$

Applying the exterior differentiation two times gives zero

$$dd = 0$$

The Hodge dual of a differential form

It is defined in d -dimensions, to take p -forms to $(d-p)$ -forms according to

$$*dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} = \frac{1}{(d-p)!}\, \epsilon^{\mu_1 \cdots \mu_p}_{\mu_{p+1} \cdots \mu_d}\, dx^{\mu_{p+1}} \wedge \ldots dx^{\mu_d}$$

For $d=4$ for example.

More generally, when the space has a metric tensor defined by

$$g$$

the Hodge dual is calculated with the formula

$$\begin{aligned} *\left(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}\right) &= \frac{1}{(n-p)!}\sqrt{g}g^{\mu_1\nu_1 \cdots \mu_p\nu_p}\varepsilon_{\nu_1 \cdots \nu_p \nu_{p+1} \cdots \nu_n} \\ &\quad \times dx^{\nu_{p+1}} \wedge \cdots \wedge dx^{\nu_n} \end{aligned}$$

There is a property of the Hodge dual

$$**\omega_p = (-1)^{p(n-p)}\,\omega_p$$

Using the Hodge dual one can define **an inner product on the space of real forms**

$$
\begin{aligned}
(\alpha_p, \beta_p) &= \int \alpha_p \wedge * \beta_p \\
&= \frac{1}{p!} \int \alpha_{\mu_1 \cdots \mu_p} \beta^{\mu_1 \cdots \mu_p} \sqrt{g} dx^1 \wedge \cdots \wedge dx^n
\end{aligned}
$$

Maxwell equations and differential forms

The gauge field

$$A = A_\mu^a \lambda^a$$

is a matrix-valued 1 form

$$A = A_\mu dx^\mu$$

called **connection 1-form**.The **field strength** tensor is the curvature two-form

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = dA + A \wedge A$$

(we can say that it is a flux through a two-dimensional surface).

The **Bianchi identity** is derived by taking the exterior derivative of the curvature

$$dF = d^2 A + dAA - AdA = FA - AF$$

which can be written by defining the covariant derivative of the field strength

$$DF \equiv dF + [A, F] = 0$$

For gauge transformations

$$g(x) : M_{2n} \to G$$

we have

$$\begin{aligned} A &\to A' = g^{-1}(A+d)\,g \\ F &\to F' = dA' + A' \wedge A' = g^{-1}Fg \end{aligned}$$

For an infinitesimal gauge transformation

$$g \approx 1 + v \text{ where } v = v^a \lambda^a$$

we have the corresponding infinitesimal transformations

$$\begin{aligned} A &\to A + dv + [A, v] = A + Dv \\ F &\to F - [v, F] \end{aligned}$$

The vector-tensor form of the Maxwell equations

$$
\begin{aligned}
J^{\beta} &= \partial_{\alpha} F^{\alpha\beta} \\
0 &= \partial_{\gamma} F_{\alpha\beta} + \partial_{\beta} F_{\gamma\alpha} + \partial_{\alpha} F_{\beta\gamma}
\end{aligned}
$$

where

$$\partial_{\mu} \equiv \left(\frac{\partial}{c\partial t}, \nabla \right)$$

$$\partial_{\alpha}\partial^{\alpha} = \text{d 'Alambertian operator}$$

$$J^{\beta} = (c\rho, \mathbf{j})$$

$$A_{\mu} = (\phi, \mathbf{A})$$

$$F^{\alpha\beta} = \partial^{\alpha} A^{\beta} - \partial^{\beta} A^{\alpha}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Using differential forms

$$dF = 0 \quad \text{(the Bianchi identity)}$$

$$d * F = *j$$

Here j is a differential *one-form* and $*j$ is a differential *three-form*, satisfying

$$d * j = 0$$

where natural units have been assumed, with $\varepsilon_0 = 1$.

The definitions

$$\begin{aligned} F &= E_x dx \wedge dt + E_y dy \wedge dt + E_z dz \wedge dt \\ &\quad + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy \end{aligned}$$

$$\begin{aligned} G &= -B_x dx \wedge dt - B_y dy \wedge dt - B_z dz \wedge dt \\ &\quad + E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy \end{aligned}$$

The sources
$$\begin{aligned} S &= j_x dy \wedge dz \wedge dt \\ &\quad + j_y dz \wedge dx \wedge dt \\ &\quad + j_z dx \wedge dy \wedge dt \\ &\quad - \rho dx \wedge dy \wedge dz \end{aligned}$$

We can say that
$$\begin{aligned} S &= j_x (*dx) + j_y (*dy) + j_z (*dz) - \rho (*dt) \\ &= *j \end{aligned}$$

The equations are
$$dF = 0$$
$$dG = -S$$

We also have
$$\frac{1}{4} \mathrm{Tr} \left(\widetilde{F}_{\mu\nu} F^{\mu\nu} \right) = \mathbf{E} \cdot \mathbf{B}$$

Fiber bundles

The official definition (Novikov)

Let (E, π, M) be a triplet where E and M are differential manifolds, $\pi : E \to M$ is a differentiable surjection and

1. for any $p \in M$ the set $E_p = \pi^{-1}(p)$, called **fiber** above p, is a vectorial space of dimension m.
2. it exists a covering by open sets $\{U_\alpha\}$ of the base M and the diffeomorphisms $G_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbf{R}^m$ and

$$G_{\alpha,p} = G_\alpha|_{E_p} : \pi^{-1}(p) = E_p \to \{p\} \times \mathbf{R}^m$$

   is an isomorphism of vectorial spaces.

Then the triplet (E, π, M) is called vectorial fiber space with base M, projection π and total space E. The space $\mathbf{R}^m$ is the fiber of (E, π, M).

The covariant derivative

Consider (Madore) a complex field that describes some physical quantity

defined over a manifold. The phase of ψ can be modified by multiplying with $g \equiv \exp(i\alpha)$ which is an element of the group $U(1)$

$$\psi = g\psi^{'}$$

However this can be done independently in every point on the basis manifold, with different values of $\alpha(x)$. Since now the phase shift introduced by these transformations would contribute to the derivation at x^{μ}, the Lagrangian expressed in terms of usual derivatives will not be left invariant by these transformations. The Lagragian remains invariant if it is expressed in terms of *covariant* derivatives, since these commutes with the transformation g:

$$D_{\mu}\left(g\psi^{'}\right) = g\left(D_{\mu}^{'}\psi^{'}\right)$$

where

$$\begin{aligned} D_\mu &= \partial_\mu + ieA_\mu \\ D_\mu^{'} &= \partial_\mu^{'} + ieA_\mu^{'} \end{aligned}$$

The invariance is obtained because the potential A_μ transforms as

$$A_\mu^{'} = A_\mu + \frac{1}{e}\partial_\mu \alpha$$

which is the usual gauge transformation for the electromagnetic potential.

The covariant derivative operators, as opposed to usual derivations, do not commute

$$[D_\mu, D_\nu] = ieF_{\mu\nu}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

This formulas are valid only if the group (here $U(1)$) is Abelian.

More generally, the transformations made in every point may be non-Abelian. Then the covariant derivatives are defined as (we suppress from the definition the factor ie)

$$D_\mu \psi = \partial_\mu \psi + [A_\mu, \psi]$$

(where we admit that also ψ is in the representation of the non-Abelian group). The transformation

$$\psi = g\psi^{'}$$

when g is from a non-Abelian group like $SU(2)$,

$$A_\mu^{'} = g^{-1} A_\mu g + g^{-1} \partial_\mu g$$

and the field tensor becomes

$$\begin{aligned} [D_\mu, D_\nu] &= F_{\mu\nu} \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \end{aligned}$$

with the transformation

$$F_{\mu\nu}^{'} = g^{-1} F_{\mu\nu} g$$

When the field is zero

$$F_{\mu\nu} \equiv 0$$

the potential takes the form of a pure gauge

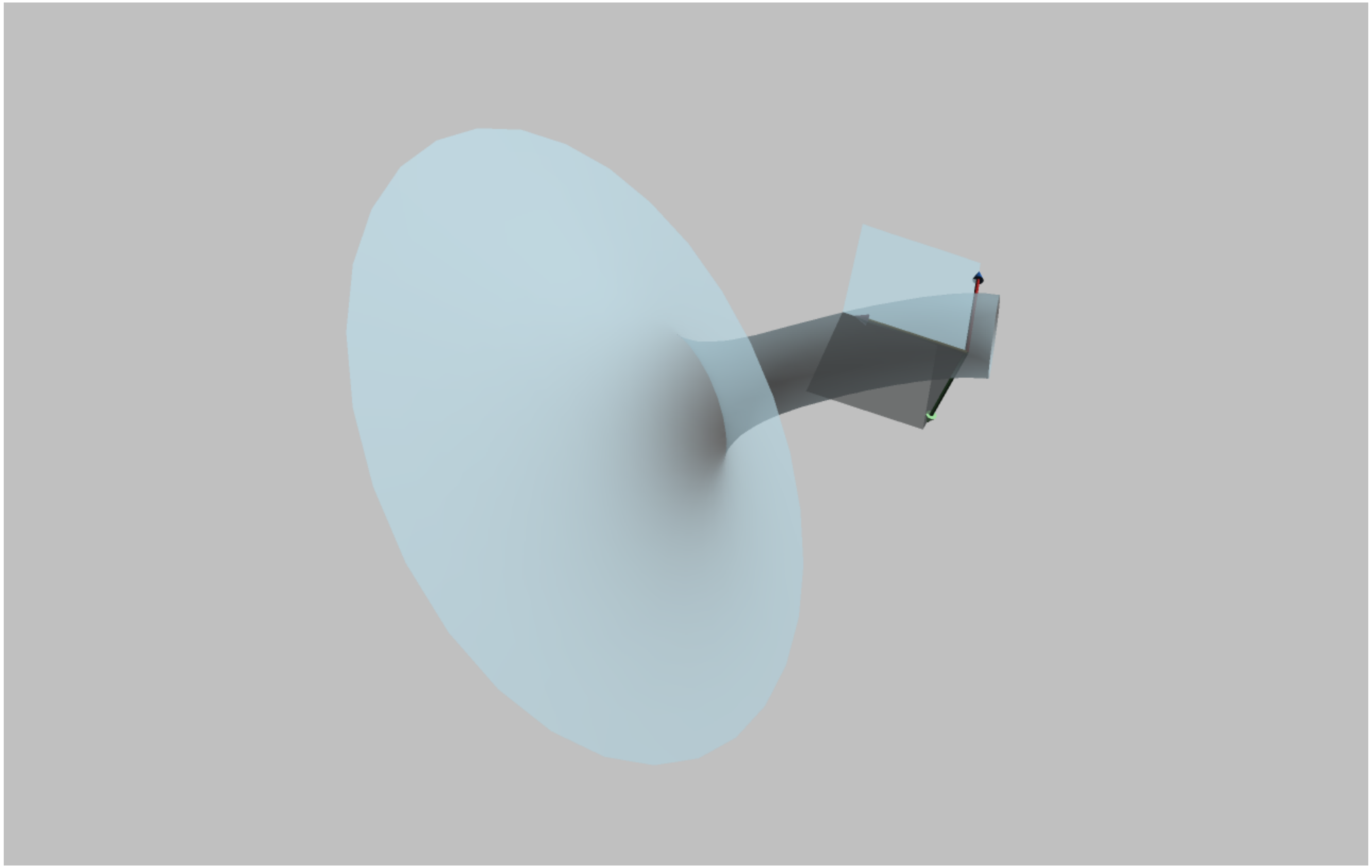
$$A_\mu = g^{-1} \partial_\mu g$$

We recognize the elements of a fiber space: the basis manifold M is the Minkowski space $\mathbf{R}^2 \times \mathbf{R}$, the fiber is the group manifold, for example $SU(2)$, the group of automorphisms of the fiber is again $SU(2)$ (principal fibration). The total space of the fiber space is locally a Cartesian product $M \times G$. Then one can define a *connection* one-form

$$\omega = g^{-1} A_\mu g dx^\mu + g^{-1} dg$$

for which the curvature two-forms is

$$\begin{aligned} \Omega &= d\omega + \omega \wedge \omega \\ &= \frac{1}{2} g^{-1} F_{\mu\nu} g dx^\mu \wedge dx^\nu \end{aligned}$$



Groups and algebras

The structures:

- group (Lie, continuous). It is in the same time a group and a manifold
- ring
- module
- algebra: the generators are the independent vectors in the tangent plane at the manifold of the group, in the idenity element

The **dimension** of a simple Lie algebra is the total *number of linearly independent generators.*

The **rank of the algebra**, r, is the maximum *number of simultaneously diagonalisable generators* of a simple Lie algebra (or the number of generators that commute between them). For example, $SU(2)$ has rank 1. And $SU(N)$ has rank $N-1$. The *rank* is the dimension of the **Cartan subalgebra**, denoted usually H.

In the Cartan-Weyl analysis the generators are written in a basis where they can be devided into two sets:

- the **Cartan subalgebra**, which is the *maximal Abelian subalgebra* of G (a maximal set of commuting generators). It contains r diagonalisable generators H_i, $i = 1, ..., r$

$$[H_i, H_j] = 0 \ , \ i, j = 1, ..., r \tag{3}$$

  This type of generators is similar to the z component of the angular momentum, J_3.

- the remaining generators of the algebra G are defined such as they satisfy the eigenvalue problems

$$[H_i, E_\mu] = \alpha_i E_\mu \ , \ i = 1, ..., r \tag{4}$$

  These generators are divided into two classes, exactly as the ladder generators.

In the case of $SU(2)$ the Cartan subalgebra has a unique element: is actually the projection of the intrinsic angular momentum along the reference axis. Elements of the Cartan subalgebra generate $U(1)$ symmetries, as rotations around these axis.

The rest of the generators of the Lie algebra, which are not in the Cartan subalgebra, are generators like the *ladder* operators, rising and lowering the angular momentum projection along a particular axis

$$J^{\pm} = \frac{1}{\sqrt{2}}\left(J_1 \pm iJ_2\right)$$

The commutations are

$$\begin{aligned}\left[J_3, J^+\right] &= J^+ \\ \left[J_3, J^-\right] &= -J^- \\ \left[J_3, J_3\right] &= 0 \\ \left[J^+, J^-\right] &= J_3\end{aligned}$$

A topological field theory: the *sigma* model

In $2D$ we have a field ϕ which is a vector of absolute magnitude 1,

attached to any point of the plane. For any $x^\mu \in \mathbf{R}^2$ we have

$$\begin{aligned} \phi &\in \mathbf{R}^3 \text{ (vector with three components)} \\ \phi^2 - 1 &= 0 \end{aligned}$$

The Lagrangian density is

$$L = \frac{1}{2} (\partial_\mu \phi) \cdot (\partial^\mu \phi)$$

the product is scalar for the components of ϕ. The model (which is the planar Heisenberg ferromagnet) consists in attaching to every point of the plane (x, y) a vector of length 1 which points to an arbitrary direction represented locally by a sphere with angle coordinates (θ, φ).

The *compactification* of the background space-time manifold.

The base space and the space of internal symmetry are now both spheres.

The field ϕ associates to any point of the compactified plane (the basis sphere) a direction in the space of internal symmetry, *i.e.* a point on a sphere. Then ϕ represents a mapping from a sphere to a sphere

$$S^2 \xrightarrow{\phi} S^2$$

The family of such mappings is divided in classes of equivalence, since a sphere can cover a sphere only an integer number of times,

$$n \in \mathbf{Z}$$

We have the structure of fiber space

$$SU(2) \sim S^2 \times U(1)$$

The curvature of this fiber bundle is

$$c_1 = -\frac{1}{8\pi} \varepsilon_{\mu\nu} \left(\phi \times \partial^\nu \phi \right) \cdot \partial^\mu \phi \; dx^1 \wedge dx^2$$

The integral of c_1, the first Chern class, on the sphere gives an integer, which is $-n$.

The action functional is written as

$$\int_{\mathbf{R}^2} \left(\partial_\mu \phi - \varepsilon_{\mu\rho} \phi \times \partial^\rho \phi\right) \left(\partial^\mu \phi - \varepsilon^{\mu\rho} \phi \times \partial_\rho \phi\right) d^2 x \geq 0$$

it follows that the action satisfies the bound condition

$$S \geq 4\pi n$$

The extremum of the action is obtained directly, without the necessity of writting the Euler-Lagrange variational equations, but simply reading off from the bound condition

$$\partial_\mu \phi - \varepsilon_{\mu\rho} \phi \times \partial^\rho \phi = 0$$

The field ϕ is self-dual.

Two solutions with winding numbers n and $n^{'}$ cannot be deformed one into the other, or, equivalently, there are infinite potential barriers separating classes of solutions.