Field theoretical methods in fluid and plasma theory

Second Part : Four Lagrangians

(Field theoretical models
for the two-D ideal incompressible Euler fluid
and for the two-D plasma in strong magnetic field)

Florin Spineanu

Association EURATOM-MEdC Romania

http://florin.spineanu.free.fr/professional/lectures/lectures.htm

1. Abelian, Chern-Simons, scalar self-interaction of fourth order. The Liouville equation. The *current density in tokamak* plasma (cf. J.B. Taylor).

2. Non-Abelian, Chern-Simons, scalar self-interaction of order four. The *sinh*-Poisson equation. The incompressible Navier-Stokes fluid in the absence of dissipation and viscosity, *i.e.* the *Euler fluid* (cf. Montgomery et al.).

3. Non-Abelian, Chern-Simons, scalar self-interaction of sixth order. An equation that seems able to describe correctly the stationary states of a *2D plasma in strong magnetic field* and of the *tropical cyclone.*

4. Abelian, Chern-Simons, scalar self-interaction of sixth order. An equation giving states or ring-type vorticity distribution, stabilized by a topological constraint. Possible physical applications: the sheared velocity layers of *H-mode* and of *Internal Transport Barriers in tokamak.*

System of interacting particles in plane

A system of particles in the plane interacting through a potential. The Hamiltonian is

$$H = \sum_{s=1}^{N} \frac{1}{2} m_s \mathbf{v}_s^2$$

where

$$m_s \mathbf{v}_s = \mathbf{p}_s - e_s \mathbf{A}\left(\mathbf{r}_s | \mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N\right)$$

the potential at the point $\mathbf{r}_s$

$$\mathbf{A}\left(\mathbf{r}_s | \mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N\right) \equiv \left(a_s^i\left(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N\right)\right)_{i=1,2}$$

$$a_s^i\left(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N\right) = \frac{1}{2\pi\kappa} \varepsilon^{ij} \sum_{q \neq s}^{N} e_q \frac{r_s^j - r_q^j}{\left|\mathbf{r}_s - \mathbf{r}_q\right|^2}$$

The vector potential $\mathbf{A}_s$ is the *curl* of the Green function of the Laplacian

$$\frac{1}{2\pi} \varepsilon^{ij} \frac{r^j}{r^2} = \varepsilon^{ij} \partial_j \frac{1}{2\pi} \ln r \qquad \nabla^2 \frac{1}{2\pi} \ln r = \delta^2(r)$$

The continuum limit is a classical field theory

- separate the matter degrees of freedom
- Consider the interaction potential as a *free* field = new degree of freedom of the system, and find the Lagrangian which can give this potential.
- Couple the matter and the field by an interaction term in the Lagrangian

According to Jackiw and Pi the field theory Lagrangian

$$L = L_{matter} + L_{CS} + L_{interaction}$$

with

$$L_{matter} = \sum_{s=1}^{N} \frac{1}{2} m_s \mathbf{v}_s^2$$

The Chern-Simons part of the Lagrangian

$$
\begin{aligned}
L_{CS} &= \frac{\kappa}{2}\int d^2r\ \varepsilon^{\alpha\beta\gamma}\partial_\alpha A_\beta A_\gamma \\
&= \frac{\kappa}{2}\int d^2r\ \frac{\partial \mathbf{A}}{\partial t}\times\mathbf{A} - \int d^2r\ A^0 B
\end{aligned}
$$

where

$$x^\mu = (ct, \mathbf{r})$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

$$\mathbf{E} = -\nabla A^0 - \frac{\partial \mathbf{A}}{\partial t}$$

The interaction Lagrangian is

$$L_{int} = \sum_{s=1}^{N} e_s \mathbf{v}_s\cdot\mathbf{A}(t,\mathbf{r}_s) - \sum_{s=1}^{N} e_s A^0(t,\mathbf{r}_s)$$

Define the current

$$v^{\mu} = (c, \mathbf{v}_s)$$

$$j^{\mu}(t, \mathbf{r}) = \sum_{s=1}^{N} e_s v_s^{\mu} \delta(\mathbf{r} - \mathbf{r}_s)$$

the interaction Lagrangian can be written

$$\begin{aligned} L_{int} &= -\int d^2r \; A_{\mu} j^{\mu} \\ &= \int d^2r \; \mathbf{A} \cdot \mathbf{j} - \int d^2r \; A^0 \rho \end{aligned}$$

The current at the continuum limit

$$j^{\mu} = (\rho, \mathbf{j})$$

with

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

Two steps to get the Hamiltonian form

1. *Eliminate the gauge-field variables in favor of the matter variables, by using the gauge-field equations of motion.*

The **equations of motion of the gauge field** are

$$\frac{\kappa}{2}\varepsilon^{\alpha\beta\gamma}F_{\alpha\beta} = j^{\mu} \quad (1)$$

$$B = -\frac{1}{\kappa}\rho$$

$$E^{i} = \frac{1}{\kappa}\varepsilon^{ij}j^{j}$$

2. *Define the canonical momenta.*

But not yet.

It is time to find the field that will represent the continuum limit of the density of discrete points

The right choice : a complex scalar field Φ.

Remember now that the momentum is the generator of the space translations which means that it has the form : $\partial / \partial x$.

(No subversive quantum activities)

Define the momenta as **covariant derivatives**

$$\begin{aligned} \mathbf{\Pi}(\mathbf{r}) &\equiv [\nabla - ie\mathbf{A}(\mathbf{r})]\Psi(\mathbf{r}) \\ &= \mathbf{D}\Psi(\mathbf{r}) \end{aligned}$$

and the conjugate

$$\mathbf{\Pi}^{\dagger} \equiv (\mathbf{D}\Psi)^{\dagger}$$

The number density operator is

$$\rho = \Psi^{\dagger}\Psi$$

The **potential** $\mathbf{A}(\mathbf{r})$ is constructed such as to solve the Chern-Simons relation between the field $\mathbf{B} = \nabla \times \mathbf{A}$ and the charge density $e\rho$:

$$B = -\frac{e}{\kappa}\rho$$

The **potential** is then

$$\mathbf{A}(\mathbf{r}) = \nabla \times \frac{e}{\kappa} \int d^2 r' \; \mathbf{G}(\mathbf{r} - \mathbf{r}') \, \rho(\mathbf{r}')$$

where $\mathbf{G}(\mathbf{r} - \mathbf{r}')$ is the Green function of the Laplaceian in plane. The *curl* of the Green function is

$$\nabla \times \mathbf{G}(\mathbf{r} - \mathbf{r}') = -\frac{1}{2\pi} \nabla \theta (\mathbf{r} - \mathbf{r}')$$

where

$$\tan \theta (\mathbf{r} - \mathbf{r}') = \frac{y - y'}{x - x'}$$

and θ is multivalued.

The Hamiltonian

$$H = \int d^2 r \; \mathcal{H}$$

is

$$\mathcal{H} = \frac{1}{2m} \left(\mathbf{D} \Psi \right)^* \left(\mathbf{D} \Psi \right) - \frac{g}{2} \left(\Psi^* \Psi \right)^2$$

with the equation of motion

$$i \frac{\partial \Psi \left(\mathbf{r}, t \right)}{\partial t} = -\frac{1}{2m} \mathbf{D}^2 \Psi \left(\mathbf{r}, t \right) + e A^0 \left(\mathbf{r}, t \right) - g \rho \left(\mathbf{r}, t \right) \Psi \left(\mathbf{r}, t \right) \qquad (2)$$

The potential is related to the density ρ and to the current $\mathbf{j}$:

$$\mathbf{A} \left(\mathbf{r}, t \right) = \nabla \times \frac{e}{\kappa} \int d^2 r \; \mathbf{G} \left(\mathbf{r} - \mathbf{r}' \right) \rho \left(\mathbf{r}', t \right) + \text{ gauge term}$$

$$A^0 \left(\mathbf{r}, t \right) = -\nabla \times \frac{e}{\kappa} \int d^2 r \; \mathbf{G} \left(\mathbf{r} - \mathbf{r}' \right) \mathbf{j} \left(\mathbf{r}', t \right) + \text{ gauge term}$$

Write Ψ as amplitude and phase $\Psi = \rho^{1/2} \exp(ie\chi)$ and inserting this expression into the equation of motion derived from the Hamiltonian the imaginary part gives the **equation of continuity**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

and the real part gives:

$$\begin{aligned} \nabla^2 \ln \rho &= 4m \left(eA^0 - g\rho \right) \\ &\quad +2 \left(e\mathbf{A} - \frac{1}{2} \nabla \times \ln \rho \right) \left(e\mathbf{A} + \frac{1}{2} \nabla \times \ln \rho \right) \end{aligned}$$

A tentative to solve for ρ and A. (*Illusory*) Skip.

The current density has the form

$$\mathbf{j} = -\frac{e}{m}\rho\mathbf{A}$$

The current $\mathbf{j}$ contains a *longitudinal part* which is due to the scalar potential, $\frac{\partial\rho}{\partial t}$ and a transversal part which is written $\nabla\times w$. **In static configurations the current is transversal** since $\frac{\partial\rho}{\partial t}$ in the continuity equation, or $\nabla\cdot\mathbf{j} = 0$.

$$A^0 = \frac{e}{\kappa}w$$

where $\mathbf{j} = \nabla\times w$. To ensure the transversality of $\mathbf{j}$ we must have

$$\mathbf{A}\cdot\nabla\rho = 0$$

which means that is has the form

$$\mathbf{A} = \nabla\times a(\rho) = (\nabla\times\rho)\, a'(\rho) \tag{3}$$

where $a(\rho)$ is a function which can be related to w using $\mathbf{j} = -\frac{e}{m}\rho\mathbf{A}$ and $\mathbf{j} = \nabla\times w$:

$$a'(\rho) = -\frac{m}{e}\frac{w'(\rho)}{\rho}$$

Then $e\mathbf{A} = -m(\nabla\times \ln\rho)\, w'(\rho)$

The **real part of the equation of motion** becomes

$$\nabla^2 \ln\rho = \frac{4e^2m}{\kappa}\left(w - \frac{\kappa g}{e^2}\rho\right) + 2m^2\left(w' + \frac{1}{2m}\right)\left(w' - \frac{1}{2m}\right)\frac{(\nabla\rho)^2}{\rho^2}$$

and **the relation between the magnetic field and the density (which solves the Chern-Simons condition)** becomes

$$\nabla^2 \ln\rho = -\frac{e^2}{m\kappa}\frac{\rho}{w'} - \frac{w''}{w'}\frac{(\nabla\rho)^2}{\rho^2}$$

The static self-dual solutions

All starts from the identity (Bogomolnyi)

$$\left|\mathbf{D}\Psi\right|^2 = \left|\left(D_1 \pm iD_2\right)\Psi\right|^2 \pm m\nabla\times\mathbf{j} \pm eB\rho$$

Then the *energy density* is

$$H = \frac{1}{2m}\left|\left(D_1 \pm iD_2\right)\Psi\right|^2 \pm \frac{1}{2}\nabla\times\mathbf{j} - \left(\frac{g}{2} \pm \frac{e^2}{2m\kappa}\right)\rho^2$$

Taking the particular relation

$$g = \mp\frac{e^2}{m\kappa}$$

and considering that the space integral of $\nabla\times\mathbf{j}$ vanishes,

$$H = \frac{1}{2m}\int d^2r\ \left|\left(D_1 \pm iD_2\right)\Psi\right|^2$$

This is non-negative and attains its minimum, zero, when Ψ

satisfies

$$D_1\Psi \pm iD_2\Psi = 0$$

or

$$\mathbf{D}\Psi = i\mathbf{D}\times\Psi$$

which is the self-duality condition.

Then decomposing again Ψ in the phase and amplitude parts,

$$\mathbf{A} = \nabla\chi \pm \frac{1}{2e}\nabla\times\ln\rho$$

Introducing in the relation derived from Chern-Simons

$$B = \nabla\times\mathbf{A} = -\frac{e}{\kappa}\rho$$

we have

$$\nabla^2\ln\rho = \pm 2\frac{e^2}{\kappa}\rho$$

which is the Liouville equation.

Second Lagrangian *Euler fluid: Non-Abelian SU* (2), *Chern-Simons, 4^{th} order*

$$\begin{aligned}\mathcal{L} &= -\varepsilon^{\mu\nu\rho} Tr\left(\partial_\mu A_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho\right) + \\ & iTr\left(\Psi^\dagger D_0 \Psi\right) - \frac{1}{2} Tr\left((D_i \Psi)^\dagger D_i \Psi\right) + \frac{1}{4} Tr\left(\left[\Psi^\dagger, \Psi\right]\right)^2\end{aligned} \tag{4}$$

where

$$D_\mu \Psi = \partial_\mu \Psi + [A_\mu, \Psi]$$

The equations of motion are

$$iD_0 \Psi = -\frac{1}{2} \mathbf{D}^2 \Psi - \frac{1}{2} \left[\left[\Psi, \Psi^\dagger\right], \Psi\right] \tag{5}$$

$$F_{\mu\nu} = -\frac{i}{2} \varepsilon_{\mu\nu\rho} J^\rho \tag{6}$$

The Hamiltonian density is

$$\mathcal{H} = \frac{1}{2} Tr\left((D_i \Psi)^\dagger (D_i \Psi) \right) - \frac{1}{4} Tr\left(\left[\Psi^\dagger, \Psi \right]^2 \right) \tag{7}$$

Using the notation $D_\pm \equiv D_1 \pm i D_2$

$$\begin{aligned} Tr\left((D_i \Psi)^\dagger (D_i \Psi) \right) &= Tr\left((D_- \Psi)^\dagger (D_- \Psi) \right) + \\ &\quad \frac{1}{2} Tr\left(\Psi^\dagger \left[\left[\Psi, \Psi^\dagger \right], \Psi \right] \right) \end{aligned}$$

Then the energy density is

$$\mathcal{H} = \frac{1}{2} Tr\left((D_- \Psi)^\dagger (D_- \Psi) \right) \geq 0 \tag{8}$$

and the Bogomol'nyi inequality is saturated at *self-duality*

$$D_- \Psi = 0 \tag{9}$$

$$\partial_+ A_- - \partial_- A_+ + [A_+, A_-] = \left[\Psi, \Psi^\dagger \right] \tag{10}$$

The *static* solutions of the *self-duality* equations : the algebraic *ansatz*:

$$A_i = \sum_{a=1}^{r} A_i^a H_a \ , \quad \Psi = \sum_{a=1}^{r} \psi^a E_a + \psi^M E_{-M}$$

$$\left[\Psi^\dagger, \Psi\right] = \sum_{a=1}^{r} |\psi^a|^2 H_a + \left|\psi^M\right|^2 H_{-M} \tag{11}$$

for $\rho_1 \equiv \left|\psi^1\right|^2$, $\rho_2 \equiv \left|\psi^{-M}\right|^2$

$$\rho_2 = \text{const } \rho_1^{-1} \tag{12}$$

$$\Delta \ln \rho_1 + 2(\rho_1 - \rho_1^{-1}) = 0 \tag{13}$$

We then have

$$\Delta\psi + \gamma \sinh(\beta\psi) = 0. \tag{14}$$

The water we drink is *self-dual*

Third Lagrangian: *2D plasma in strong magnetic field: Non-Abelian SU (2), Chern-Simons, 6^{th} order*

- gauge field, with "potential" A^{μ}, ($\mu = 0, 1, 2$ for (t, x, y)) described by the Chern-Simons Lagrangean;
- matter ("Higgs" or "scalar") field ϕ described by the covariant kinematic Lagrangean (*i.e.* covariant derivatives, implementing the minimal coupling of the gauge and matter fields)
- matter-field self-interaction given by a potential $V\left(\phi, \phi^{\dagger}\right)$ with 6^{th} power of ϕ;
- the matter and gauge fields belong to the *adjoint* representation of the algebra $SU(2)$

$$\mathcal{L} = -\kappa\varepsilon^{\mu\nu\rho}\mathrm{tr}\left(\partial_\mu A_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) \quad (15)$$
$$-\mathrm{tr}\left[\left(D^\mu\phi\right)^\dagger\left(D_\mu\phi\right)\right]$$
$$-V\left(\phi,\phi^\dagger\right)$$

Sixth order potential

$$V\left(\phi,\phi^\dagger\right) = \frac{1}{4\kappa^2}\mathrm{tr}\left[\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)^\dagger\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\right]. \quad (16)$$

The Euler Lagrange equations are

$$D_\mu D^\mu\phi = \frac{\partial V}{\partial\phi^\dagger} \quad (17)$$

$$-\kappa\varepsilon^{\nu\mu\rho}F_{\mu\rho} = iJ^\nu \quad (18)$$

The energy can be written as a sum of squares. The *self-duality* eqs.

$$
\begin{aligned}
D_{-}\phi &= 0 \\
F_{+-} &= \pm\frac{1}{\kappa^2}\left[v^2\phi - \left[\left[\phi,\phi^\dagger\right],\phi\right],\phi^\dagger\right]
\end{aligned}
\tag{19}
$$

The algebraic *ansatz* : in the Chevalley basis

$$
\begin{aligned}
\left[E_{+},E_{-}\right] &= H \\
\left[H,E_{\pm}\right] &= \pm 2E_{\pm} \\
\mathrm{tr}\left(E_{+}E_{-}\right) &= 1 \\
\mathrm{tr}\left(H^2\right) &= 2
\end{aligned}
\tag{20}
$$

The fields

$$\phi = \phi_1 E_{+} + \phi_2 E_{-}$$

$$A_{+} = aH, A_{-} = -a^* H$$

Equations for the components of the density of vorticity (here for $'+'$)

$$-\frac{1}{2}\Delta \ln \rho_1 = -\frac{1}{\kappa^2}(\rho_1 - \rho_2)\left[2(\rho_1 + \rho_2) - v^2\right] \tag{21}$$

$$-\frac{1}{2}\Delta \ln \rho_2 = \frac{1}{\kappa^2}(\rho_1 - \rho_2)\left[2(\rho_1 + \rho_2) - v^2\right] \tag{22}$$

$$\Delta \ln(\rho_1 \rho_2) = 0$$

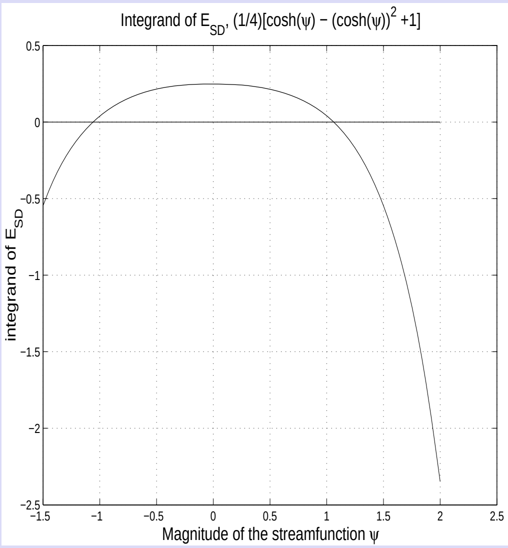
introduce a single variable

$$\rho \equiv \frac{\rho_1}{v^2/4} = \frac{v^2/4}{\rho_2} \tag{23}$$

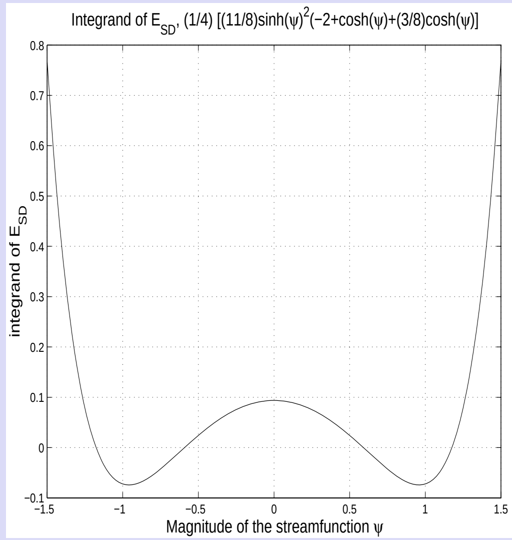
and obtain

$$-\frac{1}{2}\Delta \ln \rho = -\frac{1}{4}\left(\frac{v^2}{\kappa}\right)^2 \left(\rho - \frac{1}{\rho}\right)\left[\frac{1}{2}\left(\rho + \frac{1}{\rho}\right) - 1\right] \tag{24}$$

The energy at Self-Duality for two choices of the Bogomolnyi form for the action functional



$$\Delta\psi - \sinh\psi\,(\cosh\psi - 1) = 0$$



$$\Delta\psi + \frac{1}{2}\sinh\psi\,(\cosh\psi - 1) = 0$$

This simplest form of the equation governing the stationary states of the CHM eq.

$$\Delta\psi + \frac{1}{2}\sinh\psi\left(\cosh\psi - 1\right) = 0$$

The 'mass of the photon' is

$$m = \frac{v^2}{\kappa} = \frac{1}{\rho_s}$$

$$\begin{aligned} \kappa &\equiv c_s \\ v^2 &\equiv \Omega_{ci} \end{aligned}$$

Abelian dominance

The last Lagrangian

In certain cases the model collapses to an Abelian structure, where (ϕ, A^μ) are complex scalar functions

$$\mathcal{L} = (D^\mu \phi)^* (D_\mu \phi) + \frac{1}{4}\kappa \varepsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} - V\left(|\phi|^2\right)$$

where

$$D_\mu \phi = \frac{\partial \phi}{\partial x^\mu} + ieA_\mu \phi$$

and

$$V\left(|\phi|^2\right) = \frac{e^2}{\kappa^2} |\phi|^2 \left(|\phi|^2 - v^2\right)^2$$

with metric

$$g^{\mu\nu} = (1, -1, -1)$$

The equations of motion

$$
\begin{aligned}
D^{\mu} D_{\mu} \phi &= -\frac{\partial V}{\partial \phi^*} \\
\frac{1}{2} \varepsilon^{\mu\nu\rho} F_{\nu\rho} &= J^{\rho}
\end{aligned}
$$

where

$$J^{\mu} = ie \left[\phi^* \left(D^{\mu} \phi \right) - \left(D^{\mu} \phi \right)^* \phi \right]$$

From the second equation of motion $B = -\frac{e}{\kappa} \rho$ one finds

$$A^0 = \frac{\kappa}{2e^2} \frac{B}{|\phi|^2} - \frac{1}{e} \frac{\partial}{\partial t} \left[\text{phase of } (\phi) \right]$$

In a field theory one can obtain the energy-momentum tensor by writing the action with the explicit presence of the metric $g^{\mu\nu}$

followed by variation of the action to this metric.

$$\begin{aligned} T_{\mu\nu} &= (D_\mu\phi)^* (D_\nu\phi) + (D_\mu\phi)(D_\nu\phi)^* \\ &\quad - g_{\mu\nu}\left[(D_\lambda\phi)^* (D_\lambda\phi) - V\left(|\phi|^2\right)\right] \end{aligned}$$

The energy is the *time-time* (00) component of this tensor

$$\begin{aligned} E &= \int d^2r \left[(D_0\phi)^* (D_0\phi) + (D_k\phi)^* (D_k\phi) + V\left(|\phi|^2\right)\right] \\ &= \int d^2r \left[\left(\frac{\partial|\phi|}{\partial t}\right)^2 + \frac{\kappa^2}{4e^2}\frac{B}{|\phi|^2} + (D_k\phi)^* (D_k\phi) + V\left(|\phi|^2\right)\right] \end{aligned}$$

The second term imposes that B and $|\phi|^2$ vanish in the same points. Then the magnetic flux lies in a ring around the zeros of $|\phi|^2$.

The *SELF-DUALITY*

The energy is transformed similar to the Bogomolnyi form

$$\begin{aligned} E &= \int d^2r \left[\left| \left(D_x \pm i D_y \right) \phi \right|^2 \right. \\ &\quad + \left| \frac{\kappa}{2e} \phi^{-1} B \pm \frac{e^2}{\kappa} \phi^* \left(|\phi|^2 - v^2 \right) \right|^2 + \left. \left(\frac{\partial |\phi|}{\partial t} \right)^2 \right] \\ &\quad \pm e v^2 \Phi + \frac{1}{2} \int_{r=\infty} \mathbf{dl} \cdot \mathbf{J} \end{aligned}$$

Restrict to the states

1. static ($\partial / \partial t \equiv 0$);
2. the current goes to zero at infinity such that the last integral is zero.

Then the energy consists of a sum of squared terms plus an additional term that has a *topological* nature, proportional with the total magnetic flux through the area.

Taking to zero the squared terms we get

$$\begin{aligned} (D_x \pm i D_y)\,\phi &= 0 \\ eB &= \mp \frac{m^2}{2}\frac{|\phi|^2}{v^2}\left(1 - \frac{|\phi|^2}{v^2}\right) \end{aligned}$$

The mass parameter is

$$m \equiv 2e^2 \frac{v^2}{\kappa}$$

These are the equations of self-duality and the energy in this case is *bounded from below* by the flux

$$E \geq e v^2 \left|\Phi\right|$$

The equation for the *ring-type* vortex

The first of the two SD equations can be written

$$eA^k = \pm\varepsilon^{kj}\partial_j \ln|\phi| + \partial^k\,[\text{phase of } \phi]$$

Replacing the potential in the second SD equation we get

$$\Delta \ln\left(|\phi|^2\right) - m^2\frac{|\phi|^2}{v^2}\left(\frac{|\phi|^2}{v^2}-1\right) = 0$$

equation that is valid in points where $|\phi| \neq 0$. For these points there is an additional term, a Dirac δ coming from taking the rotational operator applied on the term containing the phase of ϕ.

$$\Delta\psi = \exp(\psi)\,[\exp(\psi)-1] + 4\pi\sum_{j=1}^{N}\delta\left(\mathbf{x}-\mathbf{x}_j\right)$$

The return of the topological constraint

At infinity ($|\phi| \simeq v$) the covariant derivative term goes to 0

$$D^k\phi \rightarrow 0 \text{ at } r \rightarrow \infty \qquad \partial_k\phi + ieA_k\phi \rightarrow 0$$

$$\int_{r=\infty} \mathbf{dl} \cdot \nabla \ln(\phi) = i\int d\,(\text{phase of } \phi) = 2\pi i n \tag{25}$$

The flux is

$$\Phi = \int d^2r\,(\nabla \times \mathbf{A}) = \frac{2\pi}{e}n$$

The magnetic flux is discrete, *integer* multiple of a physical quantity. The topological constraint is ensured by a mapping from the circle at infinity into the circle representing the space of the internal phase of the field ϕ in the asymptotic region, $S^1 \rightarrow S^1$ classified according to the first homotopy group,

$$\pi_1(S^1) = \mathbf{Z}$$