Field theoretical methods in fluid and plasma theory

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Objective: to develop a field-theoretical model that can provide a description of the $2D$ fluids close to stationarity.

The theory is relevant for the classical debate on the relation between

(1) large-scale organized, quasi-coherent flows, and

(2) "structures" (solitons, vortices, etc.)

Content

- The $2D$ discrete systems and the field theory formalism
- $2D$ water
- planetary atmosphere ($2D$ quasi-geostrophic)
  - tropical cyclone; relationships v_{θ}^{max}, R_{max}, $r_{v_{\theta}^{max}}$
- plasma (coherent) flows; crystals of vortices in non-neutral plasmas
- Related subjects: Concentration of vorticity; Contour Dynamics; statistics of turbulence; etc.

Main idea : there exist *preferred states* of the system.
The system makes transitions between these states.

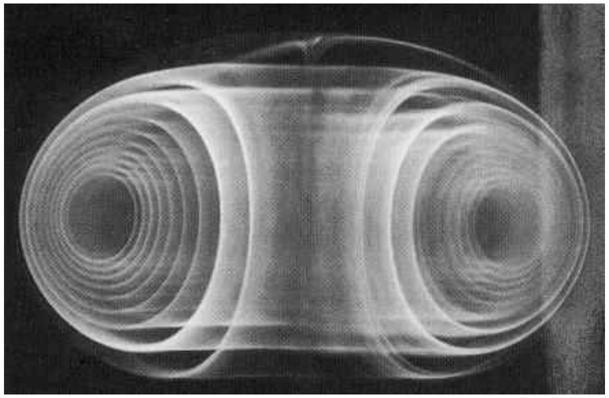
Quasi-coherent structures are observed in

- fluids (in oceans and in laboratory experiments)
- plasma (confined in strong magnetic field)
- planetary atmosphere ($2D$ quasi-geostrophic)
- non-neutral plasma (crystals of vortices)

There are common features suggesting to develop models based on the self-organization of the vorticity field. The fluids evolve at relaxation precisely to a subset of stationary states.

It is found that besides *conservation* there is also *action*

Coherent structures in fluids and plasmas (reality)

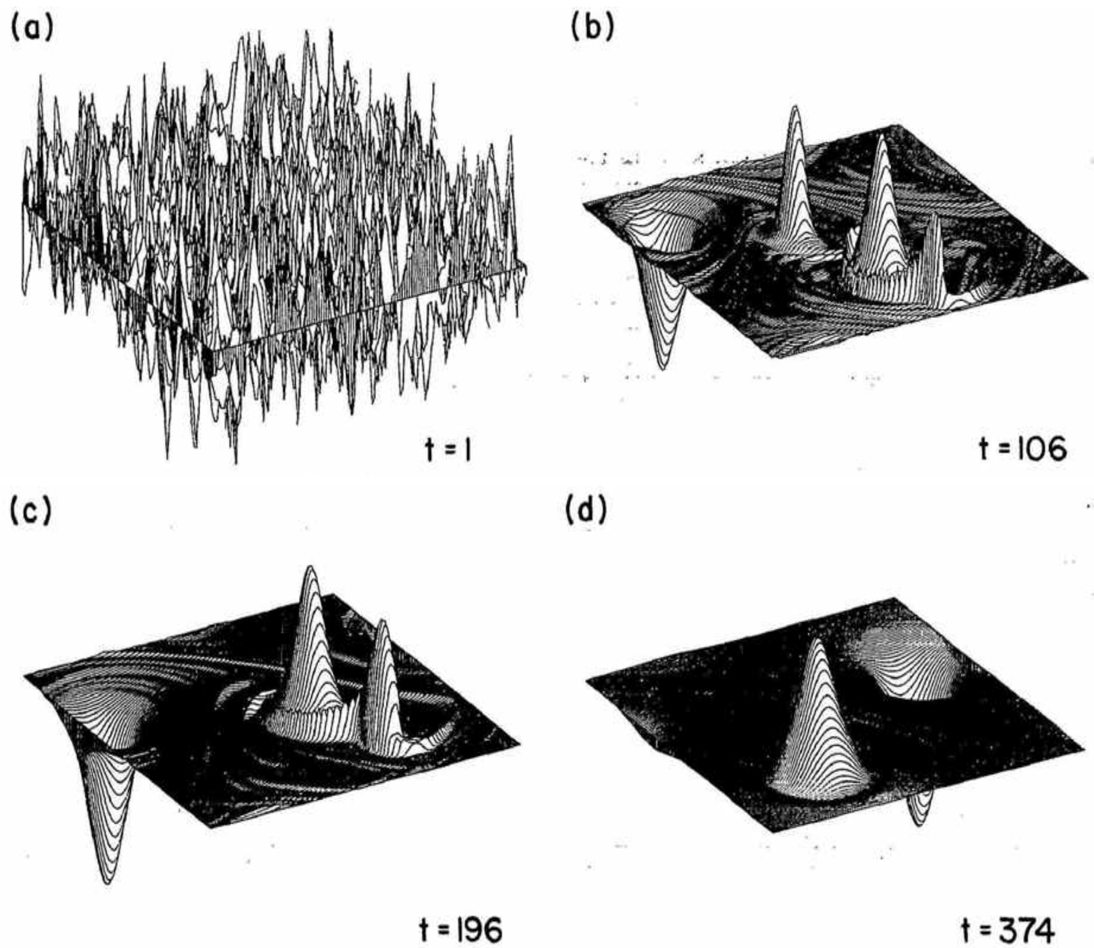


Rings of vorticity (Leonard 1998)

Nice tornado vortex.

Vortex ring emitted by the volcano Etna.

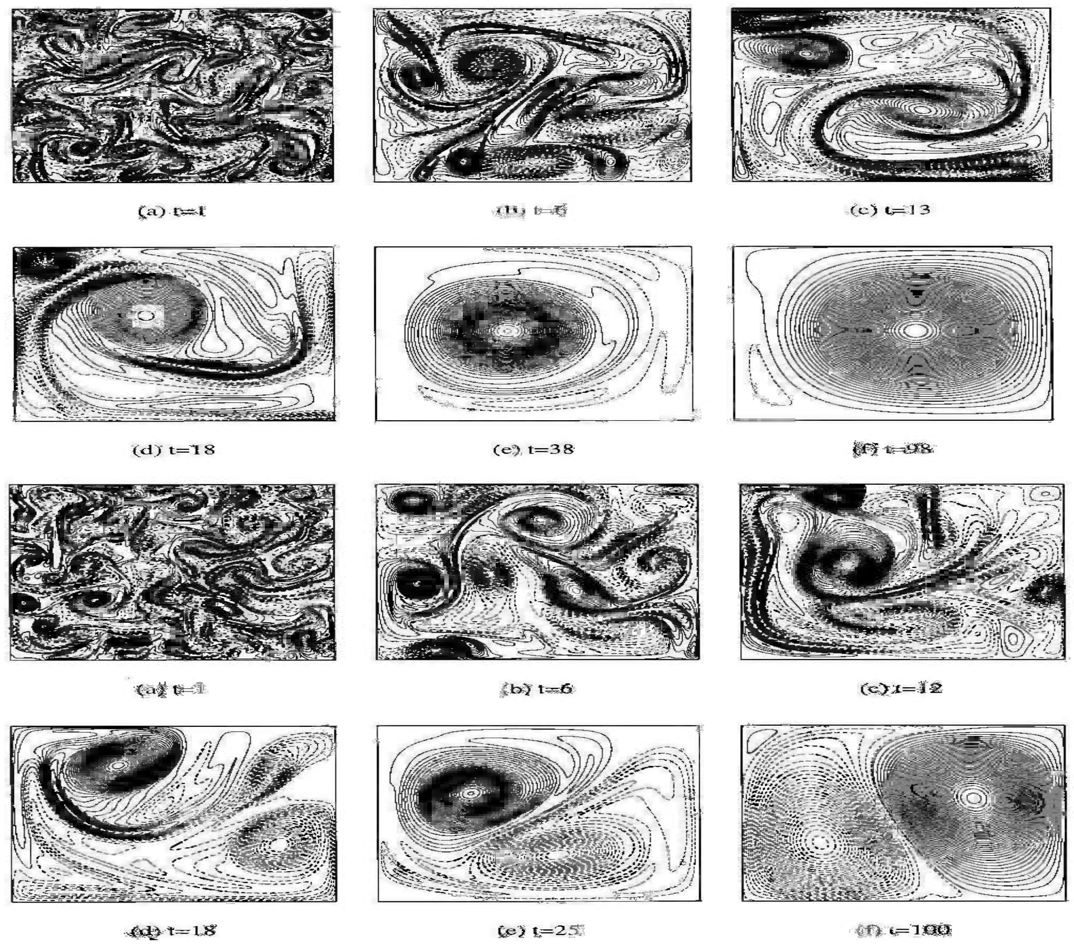
Coherent structures in fluids and plasmas (numerical 1)



Numerical simulations of the Euler equation.

D. Montgomery, W.H. Matthaeus, D. Martinez, S. Oughton, Phys. Fluids A4 (1992) 3.

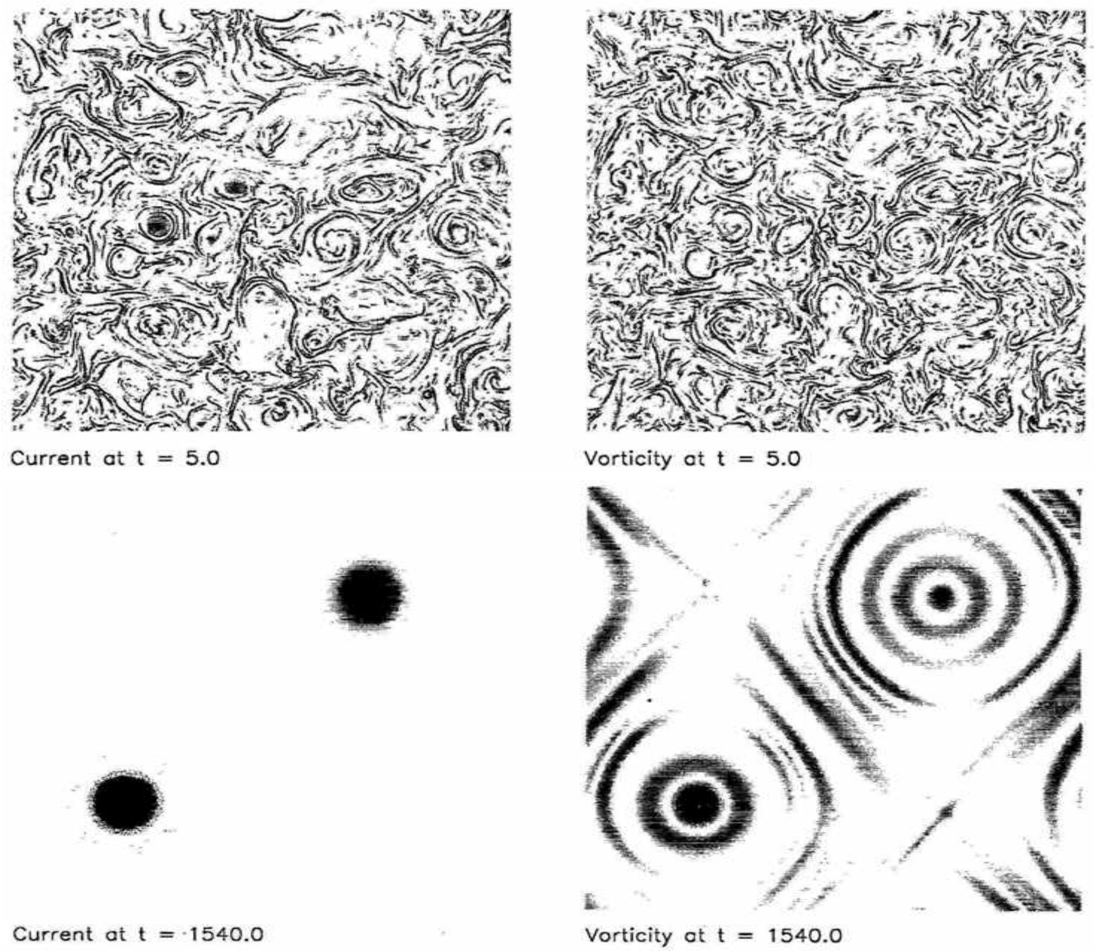
Coherent structures in fluids and plasmas (numerical 2)



Numerical simulations of the Navier-Stokes equation.

H. Brands, S. R. Maasen, H.J.H. Clercx
Phys. Rev. E 60.

Coherent structures in fluids and plasmas (numerical 3)



R. Kinney, J.C. McWilliams, T. Tajima Phys. Plasmas 2 (1995) 3623.

Numerical simulations of the MHD equations.

Compare the two approaches

Conservation eqs.

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0$$

$$mn\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} = -\nabla p - \nabla\cdot\pi + \mathbf{F}$$

$$\frac{3}{2}n\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)T = -\nabla\cdot\mathbf{q} - p\,(\nabla\cdot\mathbf{v}) - \pi : \nabla\mathbf{v} + Q$$

Valid for : coffee, ocean, sun.

Lagrangian

$$\mathcal{L}\left(x^{\mu}, \phi^{\nu}, \partial_{\rho}\phi^{\nu}\right) \rightarrow \mathcal{S} = \int dx\,dt\,\mathcal{L}$$

$$\frac{\partial}{\partial x^{\mu}}\frac{\delta\mathcal{L}}{\delta\left(\frac{\partial\phi^{\nu}}{\partial x^{\mu}}\right)} - \frac{\delta\mathcal{L}}{\delta\phi^{\nu}} = 0$$

Valid for : a single system.

Just give the initial state.

Lagrangians are preferable. But, how to find a Lagrangian ? See Phys.Rev.

Ideal fluid in $2D$ space (Euler eq.)

$$\frac{d\omega}{dt}=0\rightarrow\frac{\partial\nabla_{\perp}^{2}\psi}{\partial t}-\left[\left(-\nabla_{\perp}\psi\times\widehat{\mathbf{n}}\right)\cdot\nabla_{\perp}\right]\nabla_{\perp}^{2}\psi=0$$

At late times of the relaxation process: the *sinh*-Poisson equation

$$\Delta\psi+\gamma\sinh\left(\beta\psi\right)=0 \tag{1}$$

The Charney-Hasegawa-Mima equation

The equation (CHM) derived for the two-dimensional plasma drift waves and for Rossby waves in meteorology is:

$$\left(1-\nabla_{\perp}^{2}\right)\frac{\partial\phi}{\partial t}-\kappa\frac{\partial\phi}{\partial y}-\left[\left(-\nabla_{\perp}\phi\times\widehat{\mathbf{n}}\right)\cdot\nabla_{\perp}\right]\nabla_{\perp}^{2}\phi=0 \tag{2}$$

Equivalence with discrete models

We will try to write Lagrangians *not* directly for fluids and plasmas but for equivalent discrete models.

An equivalent discrete model for the Euler equation

$$\frac{dr_k^i}{dt} = \varepsilon^{ij} \frac{\partial}{\partial r_k^j} \sum_{n=1, n\neq k}^{N} \omega_n G\left(\mathbf{r}_k - \mathbf{r}_n\right) \ , \ i,j = 1,2 \ , \ k = 1, N \tag{1}$$

the Green function of the Laplacian

$$G\left(\mathbf{r}, \mathbf{r}'\right) \approx -\frac{1}{2\pi} \ln\left(\frac{\left|\mathbf{r} - \mathbf{r}'\right|}{L}\right) \tag{2}$$

An equivalent discrete model for the CHM equation

The equations of motion for the vortex ω_k at (x_k, y_k) under the effect of the others are

$$-2\pi\omega_k \frac{dx_k}{dt} = \frac{\partial W}{\partial y_k}$$
$$-2\pi\omega_k \frac{dy_k}{dt} = -\frac{\partial W}{\partial x_k}$$

where

$$W = \pi \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} \omega_i \omega_j K_0 \left(m \left| \mathbf{r}_i - \mathbf{r}_j \right| \right)$$

The Rosette stone,
(British Museum)

Physical model → point-like vortices → field theory.

The water Lagrangian

2D Euler fluid: Non-Abelian $SU(2)$, Chern-Simons, 4^{th} order

$$
\begin{aligned}
\mathcal{L} &= -\varepsilon^{\mu\nu\rho} Tr\left(\partial_\mu A_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho\right) + \\
& \quad iTr\left(\Psi^\dagger D_0 \Psi\right) - \frac{1}{2} Tr\left(\left(D_i \Psi\right)^\dagger D_i \Psi\right) + \frac{1}{4} Tr\left(\left[\Psi^\dagger, \Psi\right]\right)^2
\end{aligned} \tag{3}
$$

where

$$D_\mu \Psi = \partial_\mu \Psi + \left[A_\mu, \Psi\right]$$

The equations of motion are

$$iD_0 \Psi = -\frac{1}{2} \mathbf{D}^2 \Psi - \frac{1}{2}\left[\left[\Psi, \Psi^\dagger\right], \Psi\right] \tag{4}$$

$$F_{\mu\nu} = -\frac{i}{2} \varepsilon_{\mu\nu\rho} J^\rho \tag{5}$$

The Hamiltonian density is

$$\mathcal{H} = \frac{1}{2} Tr \left((D_i \Psi)^{\dagger} (D_i \Psi) \right) - \frac{1}{4} Tr \left(\left[\Psi^{\dagger}, \Psi \right]^2 \right) \tag{6}$$

Using the notation $D_{\pm} \equiv D_1 \pm i D_2$

$$\begin{aligned} Tr \left((D_i \Psi)^{\dagger} (D_i \Psi) \right) &= Tr \left((D_- \Psi)^{\dagger} (D_- \Psi) \right) + \\ &\quad \frac{1}{2} Tr \left(\Psi^{\dagger} \left[\left[\Psi, \Psi^{\dagger} \right], \Psi \right] \right) \end{aligned}$$

Then the energy density is

$$\mathcal{H} = \frac{1}{2} Tr \left((D_- \Psi)^{\dagger} (D_- \Psi) \right) \geq 0 \tag{7}$$

and the Bogomol'nyi inequality is saturated at *self-duality*

$$D_- \Psi = 0 \tag{8}$$

$$\partial_+ A_- - \partial_- A_+ + [A_+, A_-] = \left[\Psi, \Psi^{\dagger} \right] \tag{9}$$

The *static* solutions of the *self-duality* equations

The algebraic *ansatz*:

$$\begin{aligned} [E_+, E_-] &= H \\ [H, E_\pm] &= \pm 2E_\pm \\ \mathrm{tr}\,(E_+ E_-) &= 1 \\ \mathrm{tr}\,\left(H^2\right) &= 2 \end{aligned} \tag{10}$$

taking

$$\psi = \psi_1 E_+ + \psi_2 E_- \tag{11}$$

and

$$\begin{aligned} A_x &= \frac{1}{2}\left(a - a^*\right) H \\ A_y &= \frac{1}{2i}\left(a + a^*\right) H \end{aligned} \tag{12}$$

The gauge field tensor

$$F_{+-} = \left(-\partial_+ a^* - \partial_- a\right) H$$

and from the first self-duality equation

$$\frac{\partial \psi_1}{\partial x} - i \frac{\partial \psi_1}{\partial y} - 2\psi_1 a^* = 0 \tag{13}$$

$$\frac{\partial \psi_2}{\partial x} - i \frac{\partial \psi_2}{\partial y} + 2\psi_2 a^* = 0 \tag{14}$$

and their complex conjugate from $(D_- \psi)^\dagger = 0$.

Notation : $\rho_1 \equiv |\psi_1|^2$, $\rho_2 \equiv |\psi_2|^2$

$$\Delta \ln (\rho_1 \rho_2) = 0 \tag{15}$$

$$\Delta \ln \rho_1 + 2(\rho_1 - \rho_1^{-1}) = 0 \tag{16}$$

We then have

$$\Delta\psi + \gamma \sinh (\beta\psi) = 0. \tag{17}$$

The water we drink is *self-dual*

The Lagrangian of $2D$ *plasma in strong magnetic field: Non-Abelian* $SU(2)$, *Chern-Simons*, 6^{th} *order*

- gauge field, with "potential" A^{μ}, ($\mu = 0, 1, 2$ for (t, x, y)) described by the Chern-Simons Lagrangean;
- matter ("Higgs" or "scalar") field ϕ described by the covariant kinematic Lagrangean (*i.e.* covariant derivatives, implementing the minimal coupling of the gauge and matter fields)
- matter-field self-interaction given by a potential $V\left(\phi, \phi^{\dagger}\right)$ with 6^{th} power of ϕ;
- the matter and gauge fields belong to the *adjoint* representation of the algebra $SU(2)$

$$\mathcal{L} = -\kappa\varepsilon^{\mu\nu\rho}\mathrm{tr}\left(\partial_\mu A_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) \tag{18}$$
$$-\mathrm{tr}\left[\left(D^\mu\phi\right)^\dagger\left(D_\mu\phi\right)\right]$$
$$-V\left(\phi,\phi^\dagger\right)$$

Sixth order potential

$$V\left(\phi,\phi^\dagger\right) = \frac{1}{4\kappa^2}\mathrm{tr}\left[\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)^\dagger\left(\left[\left[\phi,\phi^\dagger\right],\phi\right] - v^2\phi\right)\right]. \tag{19}$$

The Euler Lagrange equations are

$$D_\mu D^\mu\phi = \frac{\partial V}{\partial\phi^\dagger} \tag{20}$$

$$-\kappa\varepsilon^{\nu\mu\rho}F_{\mu\rho} = iJ^\nu \tag{21}$$

The energy can be written as a sum of squares. The *self-duality* eqs.

$$
\begin{aligned}
D_{-}\phi &= 0 \\
F_{+-} &= \pm\frac{1}{\kappa^2}\left[v^2\phi - \left[\left[\phi,\phi^\dagger\right],\phi\right],\phi^\dagger\right]
\end{aligned}
\qquad (22)
$$

The algebraic *ansatz* : in the Chevalley basis

$$
\begin{aligned}
[E_+, E_-] &= H \\
[H, E_\pm] &= \pm 2E_\pm \\
\mathrm{tr}\,(E_+E_-) &= 1 \\
\mathrm{tr}\,\left(H^2\right) &= 2
\end{aligned}
\qquad (23)
$$

The fields

$$\phi = \phi_1 E_+ + \phi_2 E_-$$

$$A_+ = aH, A_- = -a^* H$$

Equations for the components of the density of vorticity (here for $'+'$)

$$-\frac{1}{2}\Delta\ln\rho_1 = -\frac{1}{\kappa^2}\left(\rho_1-\rho_2\right)\left[2\left(\rho_1+\rho_2\right)-v^2\right] \tag{24}$$

$$-\frac{1}{2}\Delta\ln\rho_2 = \frac{1}{\kappa^2}\left(\rho_1-\rho_2\right)\left[2\left(\rho_1+\rho_2\right)-v^2\right] \tag{25}$$

$$\Delta\ln\left(\rho_1\rho_2\right)=0$$

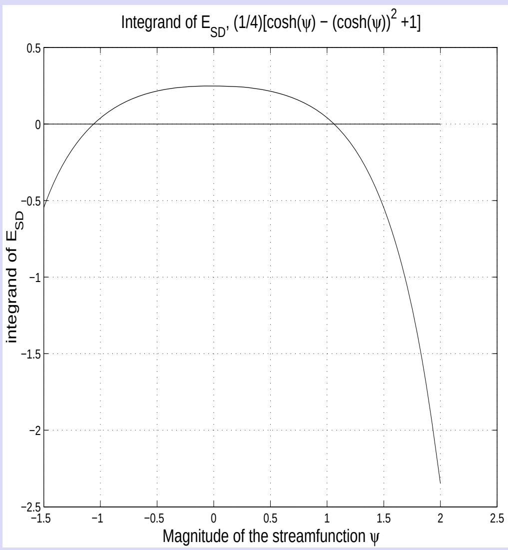
introduce a single variable

$$\rho \equiv \frac{\rho_1}{v^2/4} = \frac{v^2/4}{\rho_2} \tag{26}$$

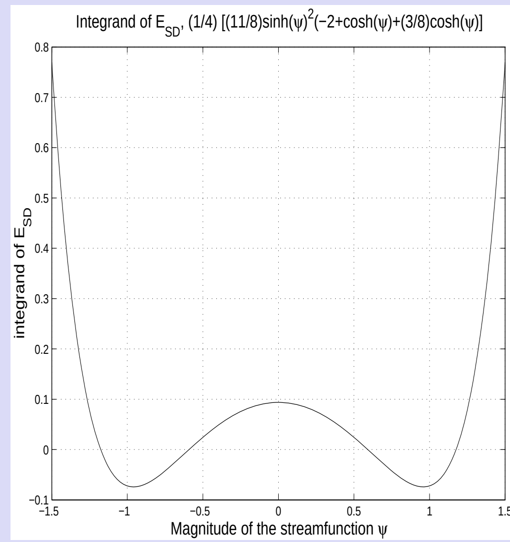
and obtain

$$-\frac{1}{2}\Delta\ln\rho = -\frac{1}{4}\left(\frac{v^2}{\kappa}\right)^2\left(\rho-\frac{1}{\rho}\right)\left[\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)-1\right] \tag{27}$$

The energy at Self-Duality for two choices of the Bogomolnyi form for the action functional



$$\Delta\psi - \sinh\psi\,(\cosh\psi - 1) = 0$$



$$\Delta\psi + \frac{1}{2}\sinh\psi\,(\cosh\psi - 1) = 0$$

This simplest form of the equation governing the stationary states of the CHM eq.

$$\Delta\psi + \frac{1}{2}\sinh\psi\,(\cosh\psi - 1) = 0$$

The 'mass of the photon' is

$$m = \frac{v^2}{\kappa} = \frac{1}{\rho_s}$$

$$\kappa \equiv c_s$$
$$v^2 \equiv \Omega_{ci}$$

Half-way Conclusions

The field theoretical formalism provides interesting results:

- identifies preferred states as extrema of an action functional
- derives explicit differential equations for these states
- allows to investigate neighboring states and reveals the existence of cuasi-degenerate directions and multiple minima of the action in the function space
- reveals the universal nature of the extrema, as self-dual states
- practical applications

The FT motel still has to be examined:

It needs a clear mapping: "formalism" - "physics"
It needs better investigation of the equations of motion
It invites to study the natural extension of the theory.

Integration of the differential equation

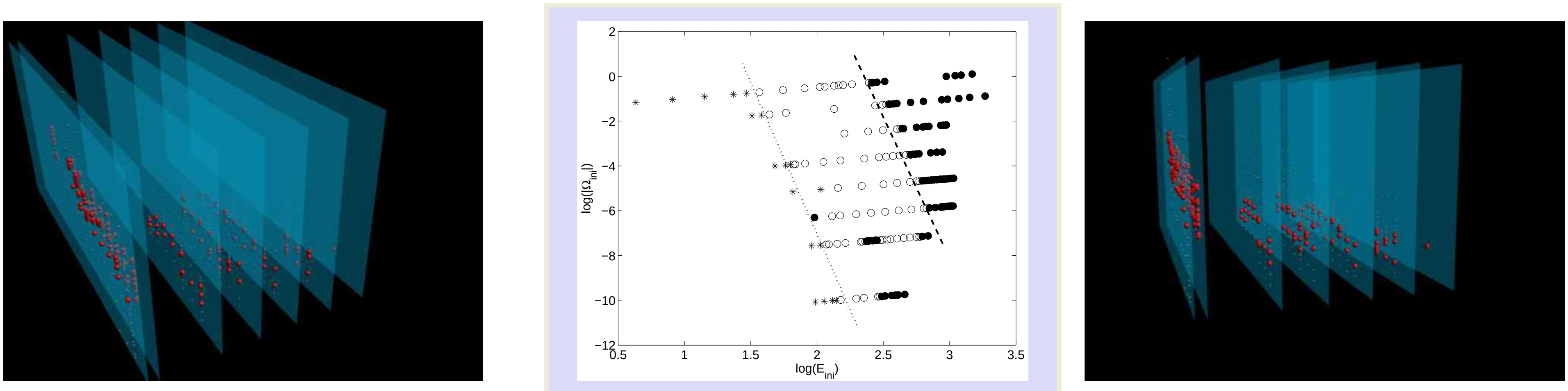
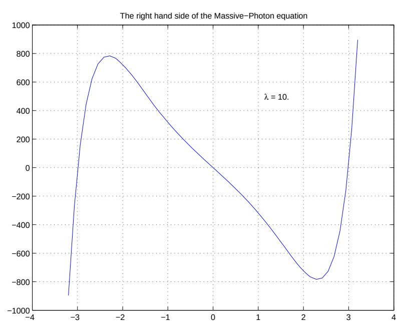


Figure 1: The structure of the space of initial conditions. The successful (coherent vortex) solutions are shown as red dots.

Solutions: trivial, turbulent, coherent, strongly concentrated.

Comparison with experiment



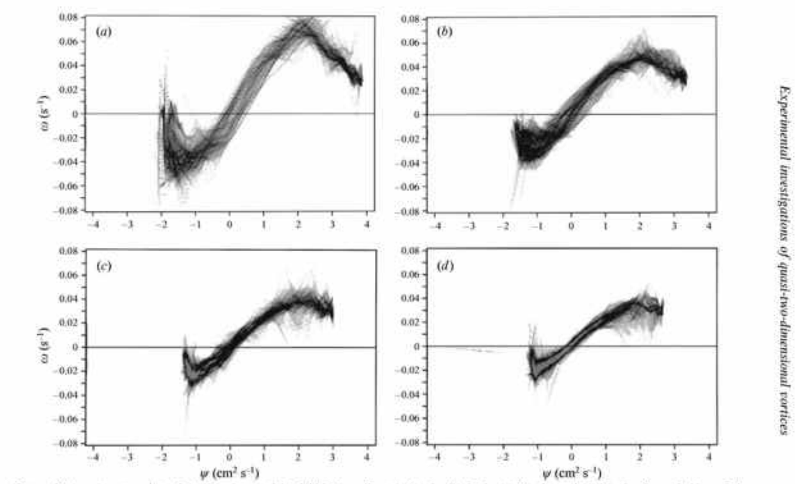
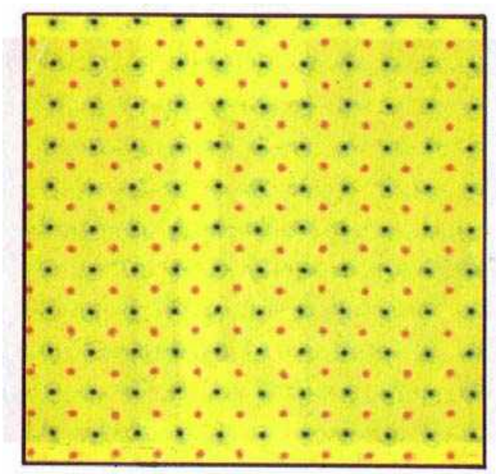


FIGURE 21. (ω, ψ) scatter plots of the decaying vorticity field in Exp. 17, at (*a*) 1 min, (*b*) 5 min, (*c*) 10 min, and (*d*) 15 min after switching off the forcing. The (ω, ψ)-values of all points on a grid in physical space, with a mesh size of approximately $(1\ mm)^2$, are plotted.

Experimental Investigation of quasi-two-dimensional in a stratified fluid with source-sink forces
Frans de Rooij, P.F. Linden, S.P. Dalziel, Journal of Fluid Mechanics **383** (1999) 249.

Figure 2: The pair (ψ, ω) and the experiment (ω is negative).

Comparison with numerical simulations in the asymptotic regime



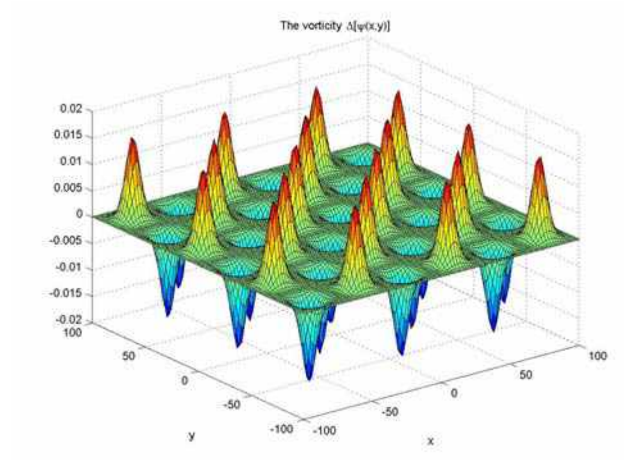


Figure 3: Comparison between numerical calculation of the CHM stationary states (Khukharin 2002) and solution of the Equation (1).

Periodic structure of vortices.

Applications

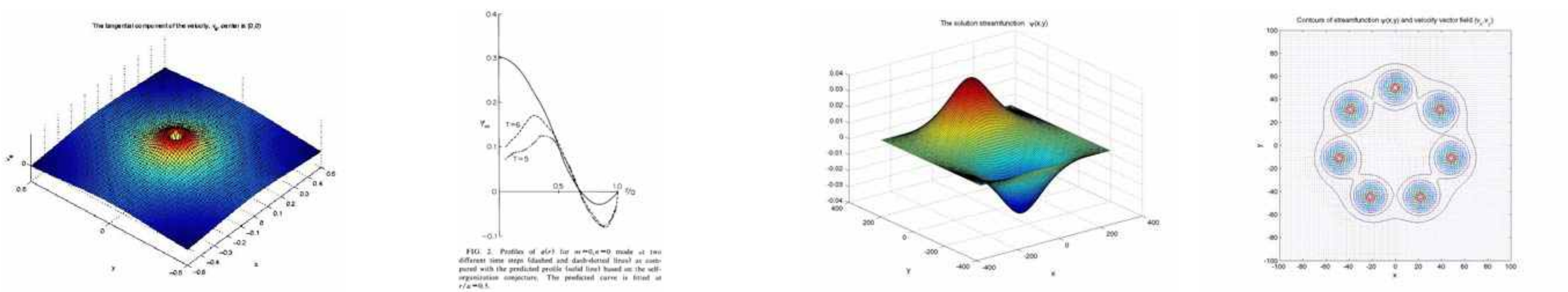
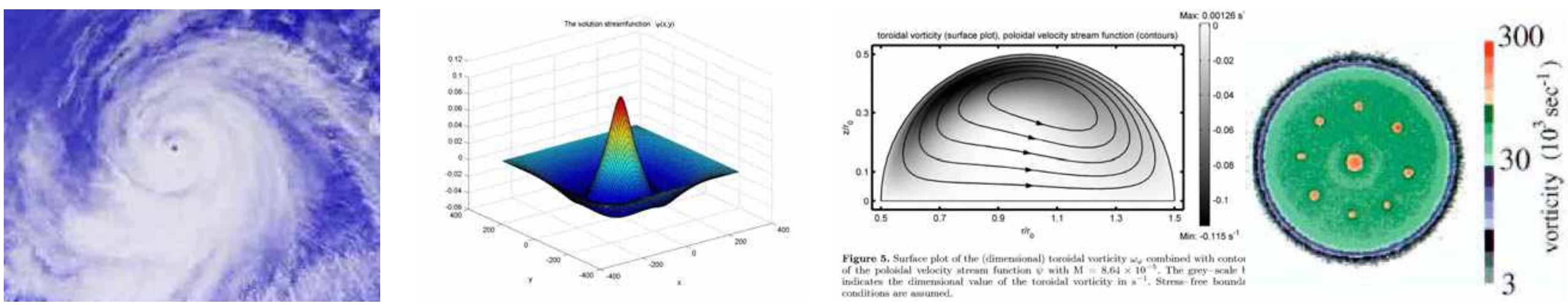


Figure 4: The atmospheric vortex, the plasma vortex, the flows in tokamak,the crystal of vortices in non-neutral plasma.



The tropical cyclone

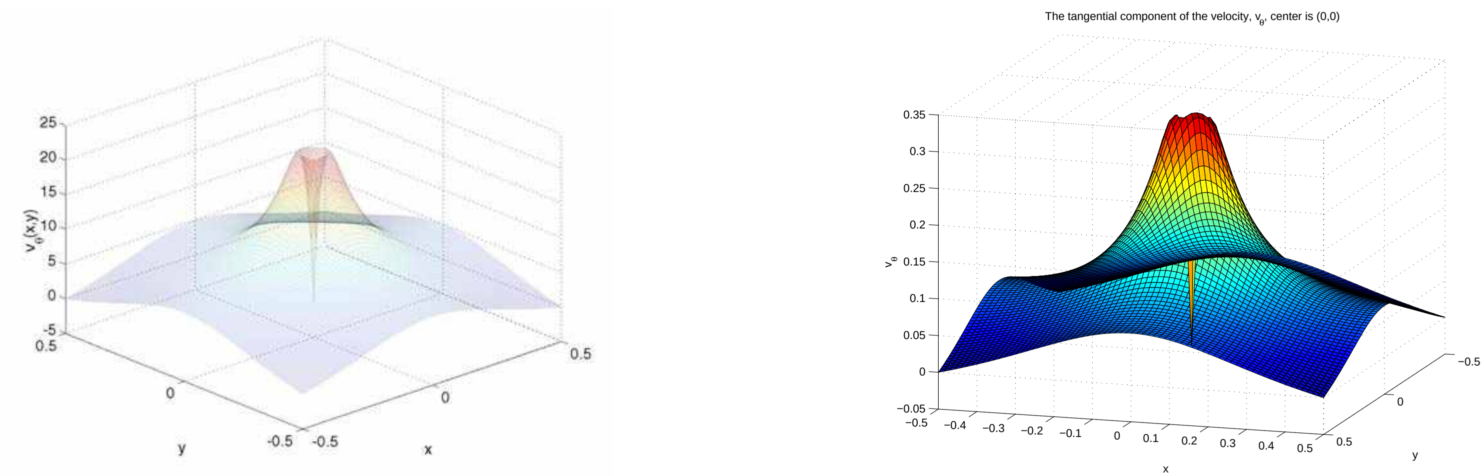
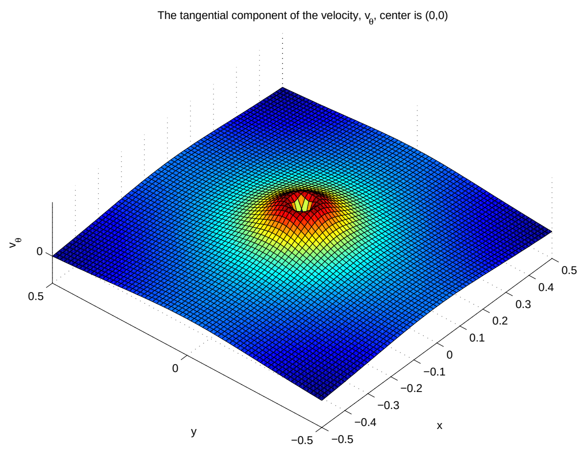


Figure 5: The tangential component of the velocity, $v_\theta(x, y)$

This is an atmospheric vortex.

The tropical cyclone , comparisons



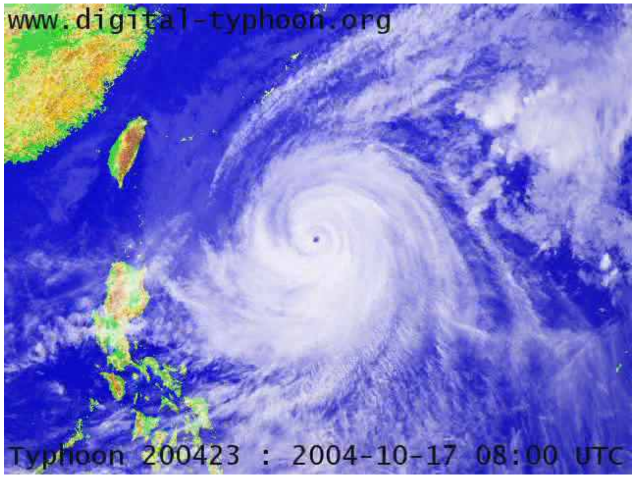
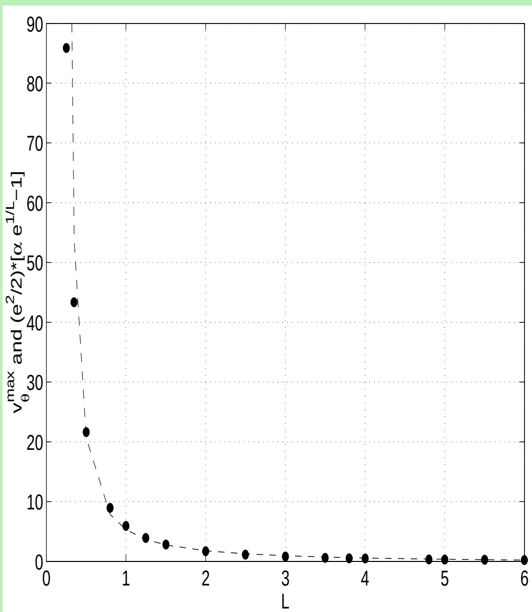


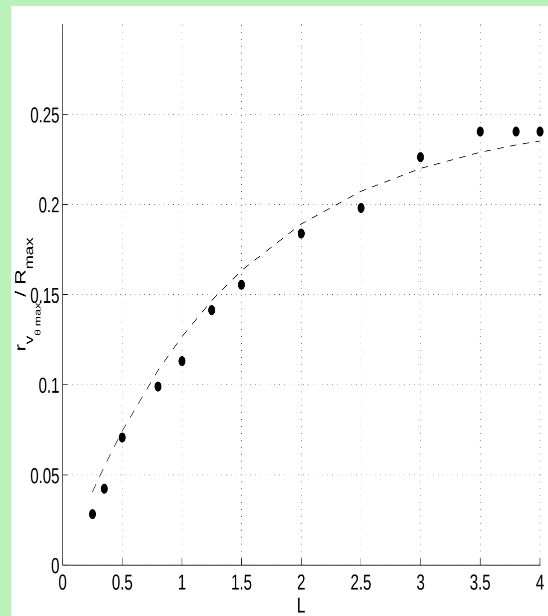
Figure 6: The solution and the image from a satelite.

The solution reproduces the *eye* radius, the radial extension and the vorticity magnitude.

Scaling relationships between main parameters of the tropical cyclone eye-wall radius, maximum tangential wind, maximum radial extension



$$v_{\theta}^{\max}(L) \simeq \frac{e^2}{2}\left[\alpha \exp\left(\frac{\sqrt{2}}{R_{\max}}\right) - 1\right]$$



$$\frac{r_{v_{\theta}max}}{R_{max}} = \frac{1}{4}\left[1 - \exp\left(-\frac{R_{max}}{2}\right)\right]$$

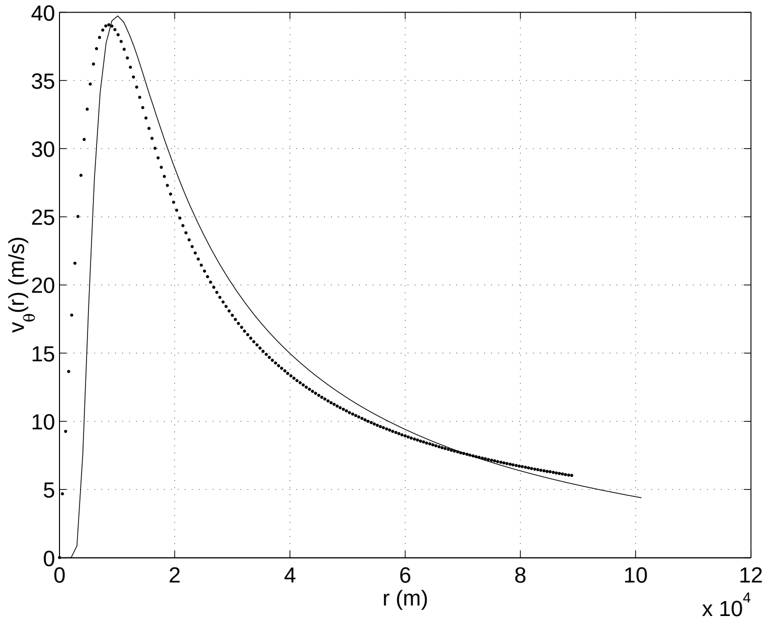
Few *remarkable* hurricanes

Table 1: Comparison between calculated and respectively observed magnitudes of the maximum tangential wind for four cases of tropical cyclones

Name	Input (obs)		Calculated			Observed
	$R_{\max}^{phys}$ (km)	$\frac{r_{v_\theta^{\max}}}{R_{\max}}$	L	Rossby ρ_g (km)	$(v_\theta^{\max})$ (m/s)	$(v_\theta^{\max})$ (m/s)
Andrew	120	0.1	0.72	117.85	64.31	68
Katrina	300	0.111	0.83	212	88.6	77.8
Rita	350	0.125	0.98	252.47	77.5	77.8
Diana	160	0.1125	0.845	133.81	56.86	55

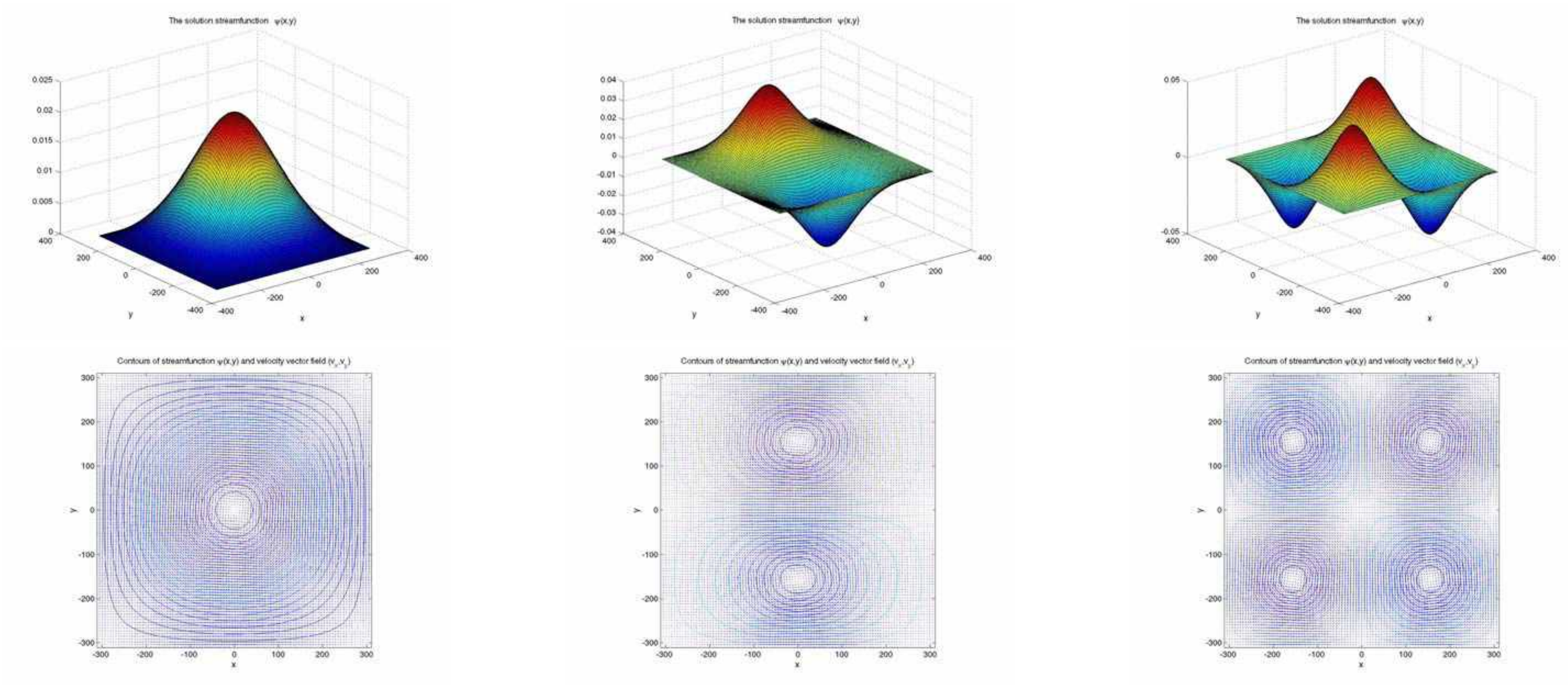
(*Comment ne pas perdre la tête?*)

Profile of the azimuthal wind velocity $v_\theta(r)$



Comparison between the Holland's empirical model for v_θ (continuous line) and our result (dotted line).

Tokamak plasma. Solution for $L = 307$: mono- and multipolar vortex



Self-organisation of the drift turbulence (Wakatani-Hasegawa)

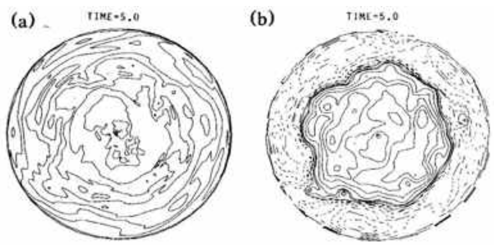


FIG. 1. (a) The density contour and (b) the potential contour from the three-dimensional computer simulation of electrostatic plasma turbulence in a cylindrical plasma with magnetic curvature and shear. In (b) the solid (dashed) lines are for the positive (negative) potential contours. Note the development of closed potential contours near the $\phi \simeq 0$ surface.

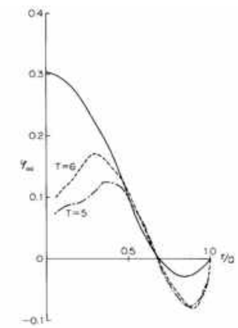
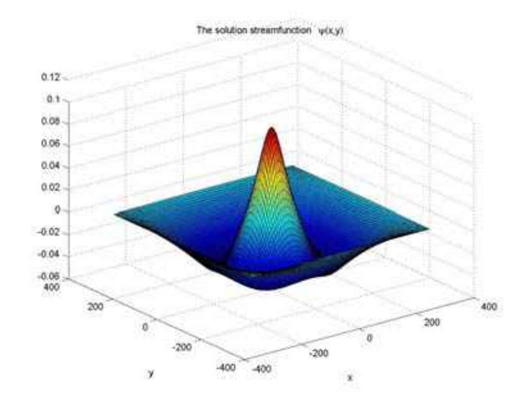
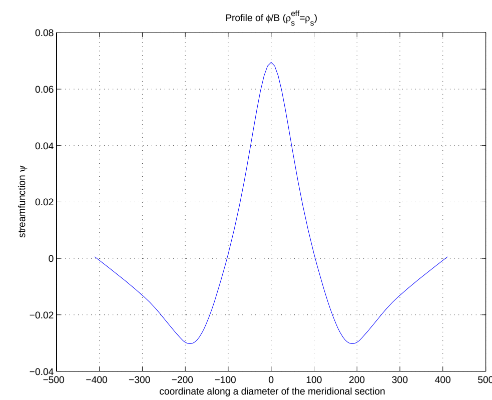


FIG. 2. Profiles of $\phi(r)$ for $m=0, n=0$ mode at two different time steps (dashed and dash-dotted lines) as compared with the predicted profile (solid line) based on the self-organization conjecture. The predicted curve is fitted at $r/a=0.5$.





The crystals of plasma vortices

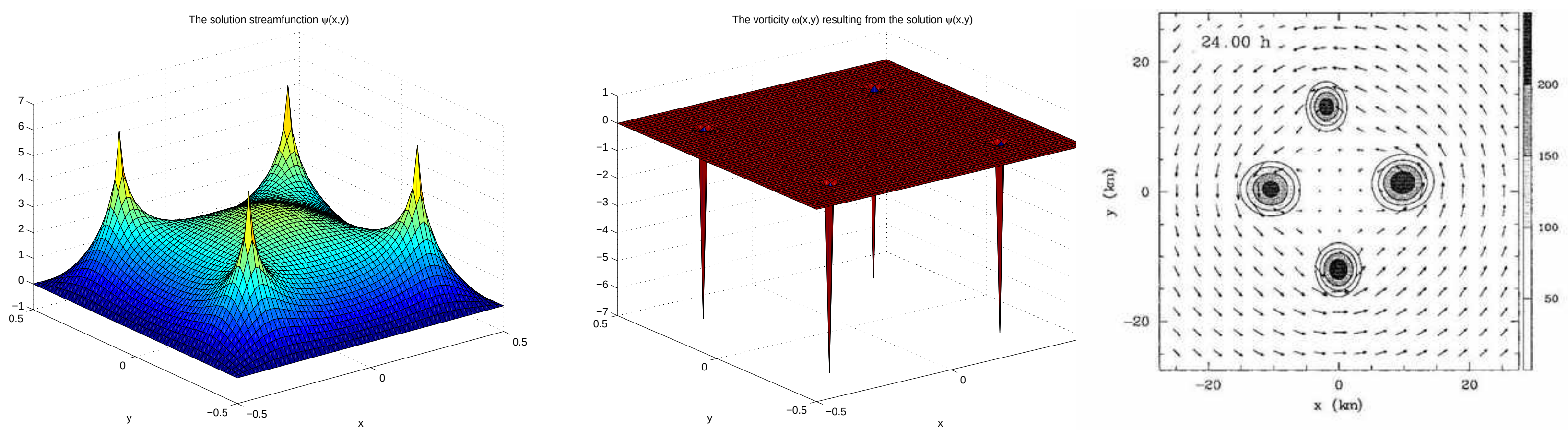


Figure 7: The crystals of plasma vortices.

Comparisons of crystal-type solutions with experiment.

Vortex crystals in non-neutral plasma

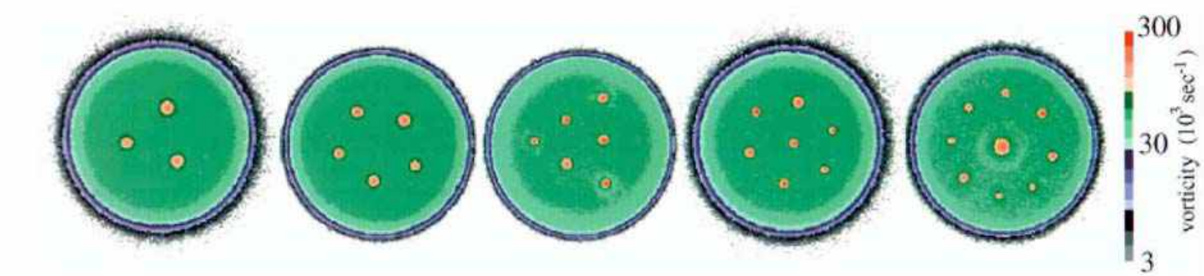
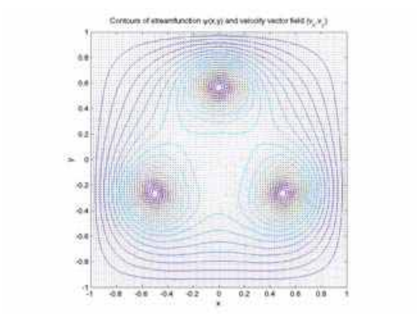
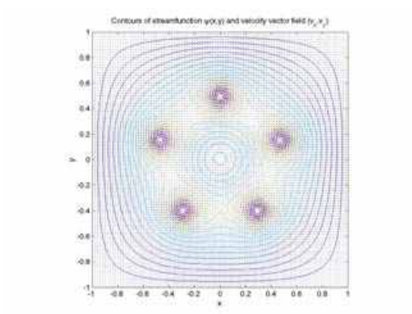
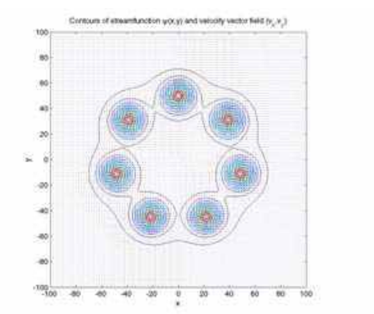


FIG. 1. Vortex crystals observed in magnetized electron columns (Ref. 8). The color map is logarithmic. This figure shows vortex crystals with (from left to right) $M=3$, 5, 6, 7, and 9 intense vortices immersed in lower vorticity backgrounds. In a vortex crystal equilibrium, the entire vorticity distribution $\zeta(r,\theta)$ is stationary in a rotating frame; i.e., ζ is a function of the variable $-\psi+\frac{1}{2}\Omega r^2$, where ψ is the stream function and Ω is the frequency of the rotating frame.







Comparison of our vortex solution with experiment.

Laser generated plasma

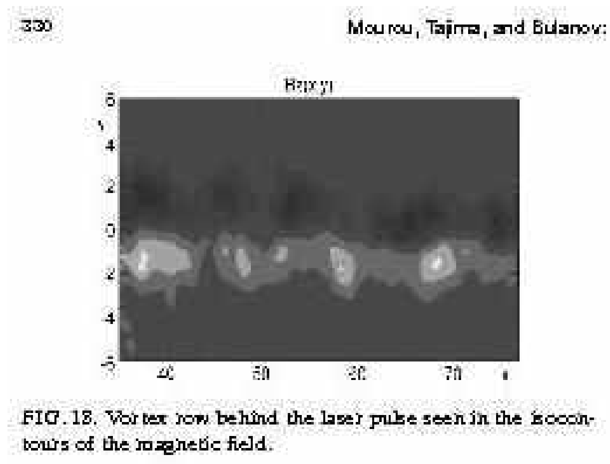


Figure 12: Structure of the magnetic field in a plasma generated by a strong Laser pulse. The equation for B has the same nonlinear form as the CHM equation.

Filamentation in Laser-generated plasmas (for Inertial Fusion)

The equation for the magnetic field generated in a laser-driven plasma

$$\begin{aligned} & \frac{1}{\mu_0 e n_0} \frac{\partial}{dt} \nabla^2 B + \left(\frac{1}{\mu_0 e n_0} \right)^2 \left[\left(-\widehat{\mathbf{e}}_z \times \nabla B \right) \cdot \nabla \right] \nabla^2 B \\ = \quad & \frac{\partial B}{\partial t} + \frac{1}{\mu_0 e n_0^2} \left[\left(-\widehat{\mathbf{e}}_z \times \nabla n_0 \right) \cdot \nabla \right] \nabla^2 n_0 \\ & + \frac{1}{e n_0} \left[\left(-\widehat{\mathbf{e}}_z \times \nabla n_0 \right) \cdot \nabla \right] \nabla^2 T_1 \end{aligned}$$

where T_1 is the perturbed temperature

$$\frac{\partial T_1}{\partial t} + T_0 \left[(\gamma - 1) \frac{n_0'}{n_0} - \frac{T_0'}{T_0} \right] \frac{\partial B}{\partial y} = - \left[\left(-\widehat{\mathbf{e}}_z \times \nabla B \right) \cdot \nabla \right] \nabla^2 T_1$$

When the T_1 perturbation and the *scalar nonlinearity* $B \partial B / \partial y$ can be neglected, the equation for B becomes the classical CHM-type equation.

A sheet of current is broken up into filaments.

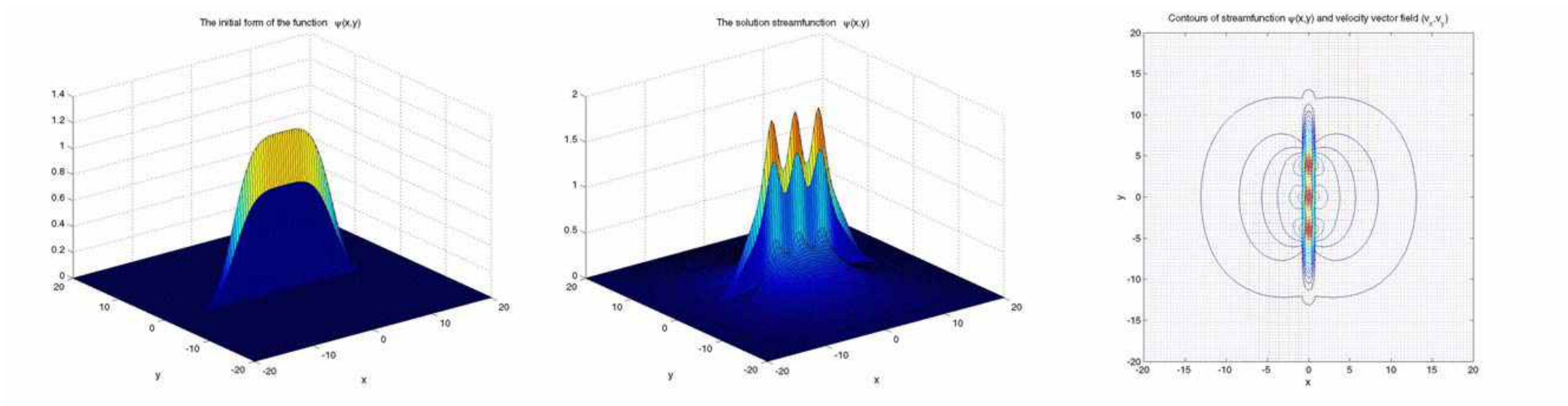


Figure 8: Filamentation in a current sheet.

Current sheet.

Final Conclusions

It seems it works.

- Hamlet: *The rest is silance*
- Einsetin: *The rest is details*
- (Field Theory Book): *The rest is gauge*

Save you work.

Prepare for shutdown.

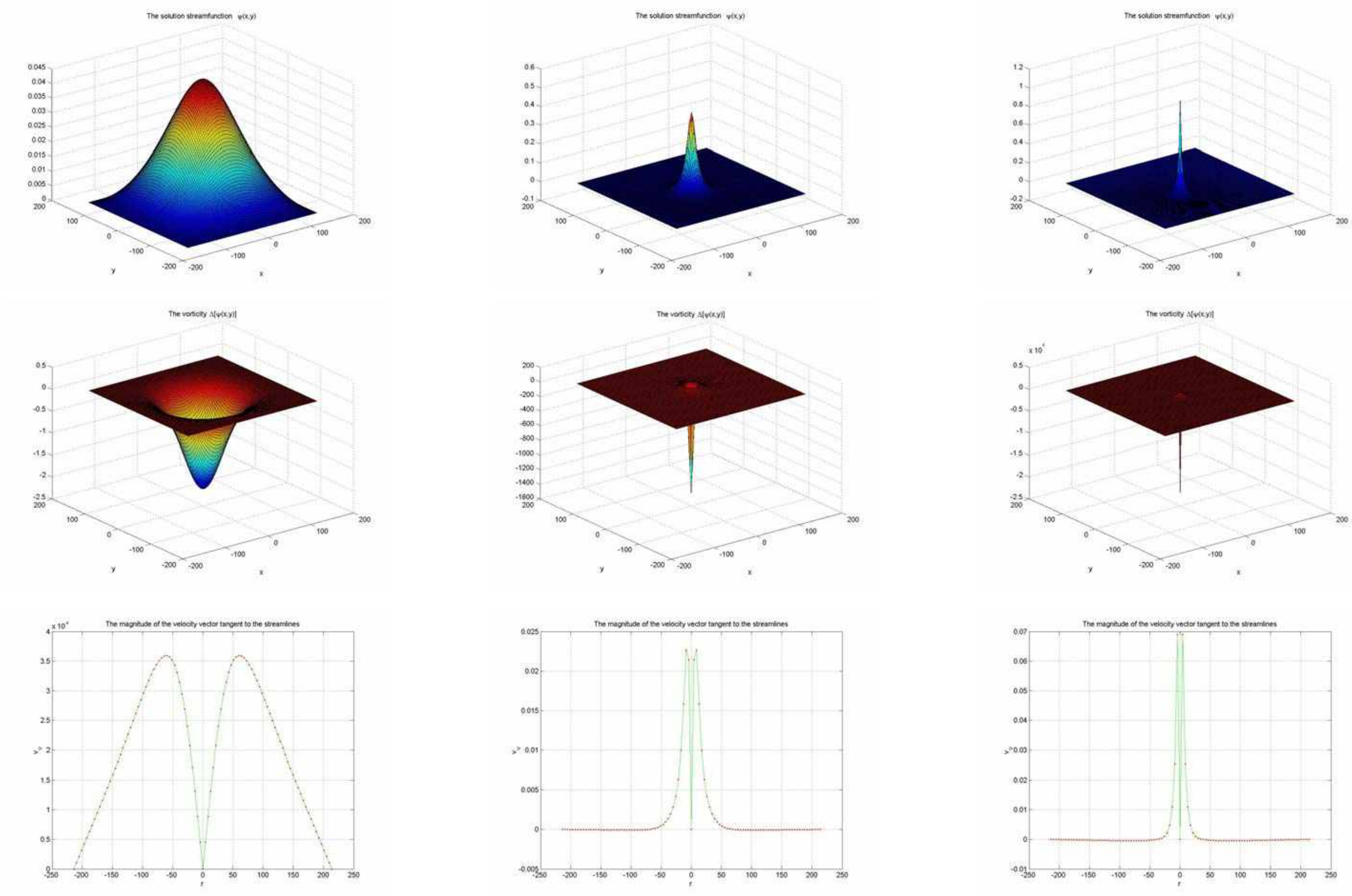
Quasi-degenerate directions in the function space of solutions

There is a class of functions that verify to a good precision the equation but they are NOT exact solutions.

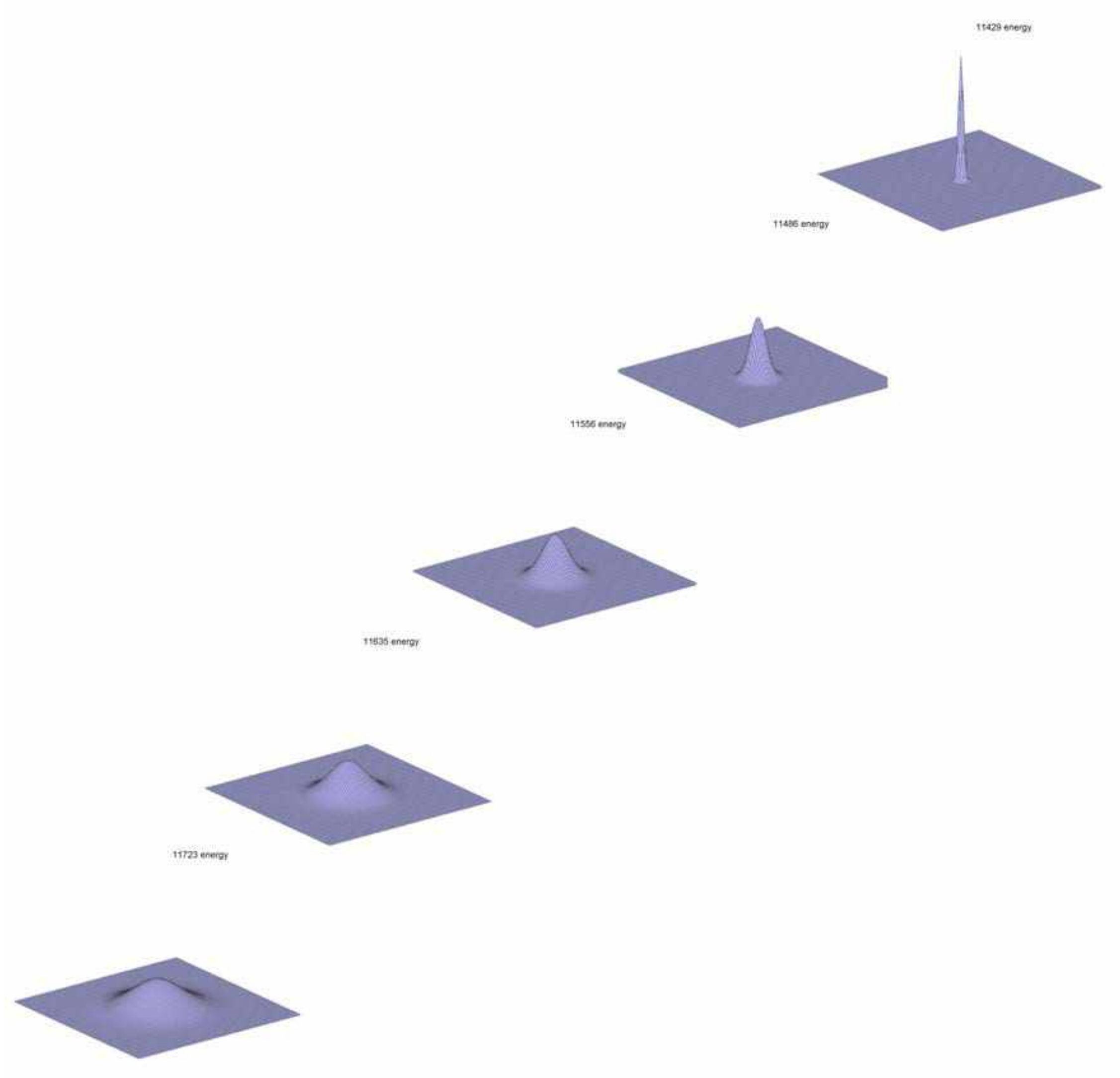
Solutions and approx-solutions, representing static flow configurations, may have:

- different shapes (from smooth to highly concentrated)
- approximately the same energy and total vorticity

This suggests that the system may *slide* along paths in the function space at almost no cost in energy or vorticity. This is interesting for *vorticity concentration*


The solution streamfunction ψ(x,y)
The solution streamfunction ψ(x,y)
The solution streamfunction ψ(x,y)
The vorticity Δ[ψ(x,y)]
The vorticity Δ[ψ(x,y)]
The vorticity Δ[ψ(x,y)]
The magnitude of the velocity vector tangent to the streamlines
The magnitude of the velocity vector tangent to the streamlines
The magnitude of the velocity vector tangent to the streamlines

Peaked profiles have lower energy



Numerical solution starting with *sech4/3*

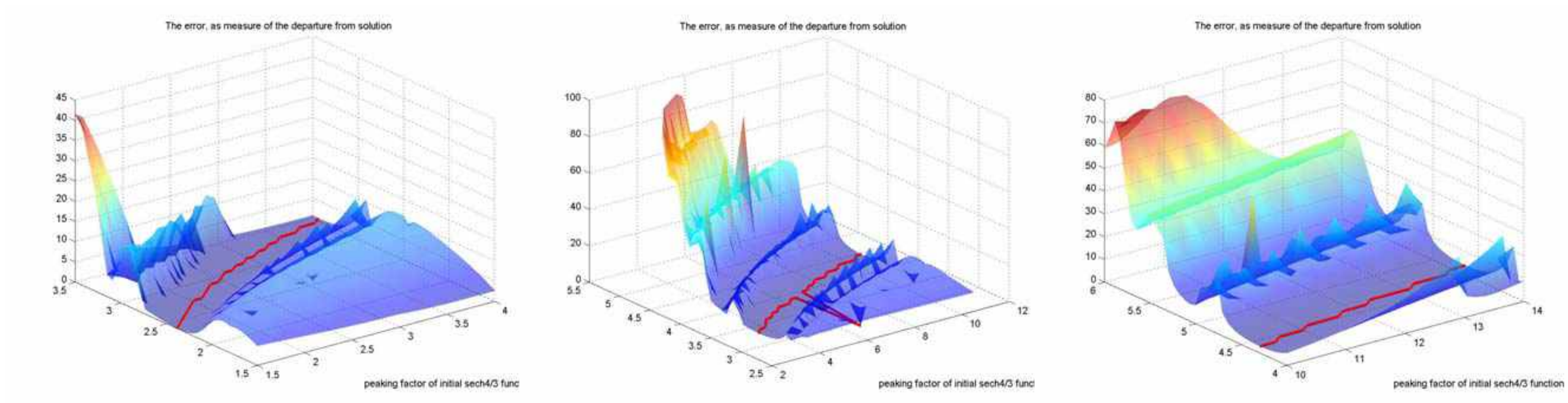


Figure 9: Three intervals on the (peaking factor, amplitude) parameter space.

Very weak variation of the *error functional* along the path (line of minimum error relative to the exact solution).

Radial integration

Figure 10: The functional error $\int d^2r(\omega + nl)^2$.

String of quasi-solutions.

Along the string of quasi-solutions the vortices are more and more concentrated

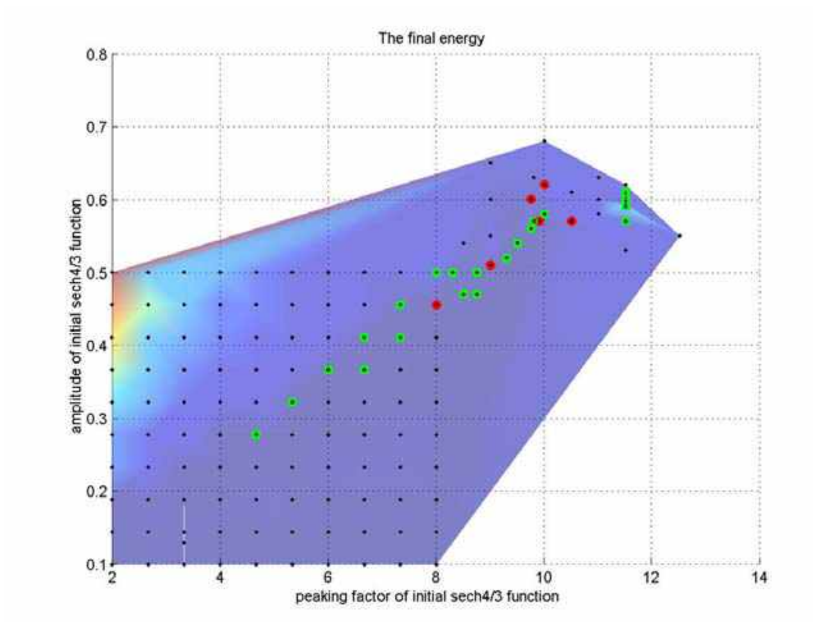


Figure 11: Green points: smooth, but progressively more peaked vortices; red: quasi-singular vortices.

The energies $\mathcal{E}_{final}$ and the vorticities Ω_{final} are only slightly different. We conclude that the system can drift along this path, under the action of even a small external drive.

System of interacting particles in plane

A system of particles in the plane interacting through a potential. The Hamiltonian is

$$H = \sum_{s=1}^{N} \frac{1}{2} m_s \mathbf{v}_s^2$$

where

$$m_s \mathbf{v}_s = \mathbf{p}_s - e_s \mathbf{A} \left(\mathbf{r}_s | \mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N \right)$$

the potential at the point $\mathbf{r}_s$

$$\mathbf{A} \left(\mathbf{r}_s | \mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N \right) \equiv \left(a_s^i \left(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N \right) \right)_{i=1,2}$$

$$a_s^i \left(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N \right) = \frac{1}{2\pi\kappa} \varepsilon^{ij} \sum_{q \neq s}^{N} e_q \frac{r_s^j - r_q^j}{\left| \mathbf{r}_s - \mathbf{r}_q \right|^2}$$

The vector potential $\mathbf{A}_s$ is the *curl* of the Green function of the Laplacian

$$\frac{1}{2\pi} \varepsilon^{ij} \frac{r^j}{r^2} = \varepsilon^{ij} \partial_j \frac{1}{2\pi} \ln r \qquad\qquad \nabla^2 \frac{1}{2\pi} \ln r = \delta^2 \left(r \right)$$

The continuum limit is a classical field theory

- separate the matter degrees of freedom
- Consider the interaction potential as a *free* field = new degree of freedom of the system, and find the Lagrangian which can give this potential.
- Couple the matter and the field by an interaction term in the Lagrangian

According to Jackiw and Pi the field theory Lagrangian

$$L = L_{matter} + L_{CS} + L_{interaction}$$

with

$$L_{matter} = \sum_{s=1}^{N} \frac{1}{2} m_s \mathbf{v}_s^2$$

The Chern-Simons part of the Lagrangian

$$\begin{aligned} L_{CS} &= \frac{\kappa}{2}\int d^2r\,\varepsilon^{\alpha\beta\gamma}\partial_\alpha A_\beta A_\gamma \\ &= \frac{\kappa}{2}\int d^2r\,\frac{\partial \mathbf{A}}{\partial t}\times\mathbf{A} - \int d^2r\,A^0 B \end{aligned}$$

where

$$x^\mu = (ct, \mathbf{r})$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

$$\mathbf{E} = -\nabla A^0 - \frac{\partial \mathbf{A}}{\partial t}$$

The interaction Lagrangian is

$$L_{int} = \sum_{s=1}^{N} e_s \mathbf{v}_s \cdot \mathbf{A}(t, \mathbf{r}_s) - \sum_{s=1}^{N} e_s A^0 (t, \mathbf{r}_s)$$

Define the current

$$v^{\mu} = (c, \mathbf{v}_s)$$

$$j^{\mu}(t, \mathbf{r}) = \sum_{s=1}^{N} e_s v_s^{\mu} \delta(\mathbf{r} - \mathbf{r}_s)$$

the interaction Lagrangian can be written

$$\begin{aligned} L_{int} &= -\int d^2r \; A_{\mu} j^{\mu} \\ &= \int d^2r \; \mathbf{A} \cdot \mathbf{j} - \int d^2r \; A^0 \rho \end{aligned}$$

The current at the continuum limit

$$j^{\mu} = (\rho, \mathbf{j})$$

with

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

Two steps to get the Hamiltonian form

1. *Eliminate the gauge-field variables in favor of the matter variables, by using the gauge-field equations of motion.*

The **equations of motion of the gauge field** are

$$\frac{\kappa}{2}\varepsilon^{\alpha\beta\gamma}F_{\alpha\beta} = j^{\mu} \tag{28}$$

$$B = -\frac{1}{\kappa}\rho$$

$$E^{i} = \frac{1}{\kappa}\varepsilon^{ij}j^{j}$$

2. *Define the canonical momenta.*

But not yet.

It is time to find the field that will represent the continuum limit of the density of discrete points

The right choice : a complex scalar field Φ.

Remember now that the momentum is the generator of the space translations which means that it has the form : $\partial/\partial x$.

(No subversive quantum activities)

Define the momenta as **covariant derivatives**

$$\begin{aligned}\mathbf{\Pi}(\mathbf{r}) &\equiv [\nabla - ie\mathbf{A}(\mathbf{r})]\,\Psi(\mathbf{r}) \\ &= \mathbf{D}\Psi(\mathbf{r})\end{aligned}$$

and the conjugate

$$\mathbf{\Pi}^{\dagger} \equiv (\mathbf{D}\Psi)^{\dagger}$$

The number density operator is

$$\rho = \Psi^{\dagger}\Psi$$

The **potential** $\mathbf{A}(\mathbf{r})$ is constructed such as to solve the Chern-Simons relation between the field $\mathbf{B} = \nabla \times \mathbf{A}$ and the charge density $e\rho$:

$$B = -\frac{e}{\kappa}\rho$$

The **potential** is then

$$\mathbf{A}(\mathbf{r}) = \nabla \times \frac{e}{\kappa} \int d^2r' \, \mathbf{G}(\mathbf{r} - \mathbf{r}') \, \rho(\mathbf{r}')$$

where $\mathbf{G}(\mathbf{r} - \mathbf{r}')$ is the Green function of the Laplaceian in plane. The *curl* of the Green function is

$$\nabla \times \mathbf{G}(\mathbf{r} - \mathbf{r}') = -\frac{1}{2\pi} \nabla \theta (\mathbf{r} - \mathbf{r}')$$

where

$$\tan \theta (\mathbf{r} - \mathbf{r}') = \frac{y - y'}{x - x'}$$

and θ is multivalued.

The Hamiltonian

$$H = \int d^2 r \; H$$

is

$$H = \frac{1}{2m} \left(\mathbf{D}\Psi \right)^* \left(\mathbf{D}\Psi \right) - \frac{g}{2} \left(\Psi^* \Psi \right)^2$$

with the equation of motion

$$i \frac{\partial \Psi \left(\mathbf{r}, t \right)}{\partial t} = -\frac{1}{2m} \mathbf{D}^2 \Psi \left(\mathbf{r}, t \right) + e A^0 \left(\mathbf{r}, t \right) - g \rho \left(\mathbf{r}, t \right) \Psi \left(\mathbf{r}, t \right) \tag{29}$$

The potential is related to the density ρ and to the current $\mathbf{j}$:

$$\mathbf{A} \left(\mathbf{r}, t \right) = \nabla \times \frac{e}{\kappa} \int d^2 r \; \mathbf{G} \left(\mathbf{r} - \mathbf{r}' \right) \rho \left(\mathbf{r}', t \right) + \text{ gauge term}$$

$$A^0 \left(\mathbf{r}, t \right) = -\nabla \times \frac{e}{\kappa} \int d^2 r \; \mathbf{G} \left(\mathbf{r} - \mathbf{r}' \right) \mathbf{j} \left(\mathbf{r}', t \right) + \text{ gauge term}$$

Write Ψ as amplitude and phase $\Psi = \rho^{1/2} \exp(ie\chi)$ and inserting this expression into the equation of motion derived from the Hamiltonian the imaginary part gives the **equation of continuity**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

and the real part gives:

$$\begin{aligned} \nabla^2 \ln \rho &= 4m \left(eA^0 - g\rho\right) \\ &\quad + 2\left(e\mathbf{A} - \frac{1}{2}\nabla \times \ln \rho\right)\left(e\mathbf{A} + \frac{1}{2}\nabla \times \ln \rho\right) \end{aligned}$$

The static self-dual solutions

All starts from the identity (Bogomolnyi)

$$|\mathbf{D}\Psi|^2 = |(D_1 \pm iD_2)\,\Psi|^2 \pm m\nabla\times\mathbf{j} \pm eB\rho$$

Then the *energy density* is

$$H = \frac{1}{2m}\,|(D_1 \pm iD_2)\,\Psi|^2 \pm \frac{1}{2}\nabla\times\mathbf{j} - \left(\frac{g}{2} \pm \frac{e^2}{2m\kappa}\right)\rho^2$$

Taking the particular relation

$$g = \mp\frac{e^2}{m\kappa}$$

and considering that the space integral of $\nabla\times\mathbf{j}$ vanishes,

$$H = \frac{1}{2m}\int d^2r\ \ |(D_1 \pm iD_2)\,\Psi|^2$$

This is non-negative and attains its minimum, zero, when Ψ

satisfies

$$D_1\Psi \pm iD_2\Psi = 0$$

or

$$\mathbf{D}\Psi = i\mathbf{D}\times\Psi$$

which is the self-duality condition.

Then decomposing again Ψ in the phase and amplitude parts,

$$\mathbf{A} = \nabla\chi \pm \frac{1}{2e}\nabla\times\ln\rho$$

Introducing in the relation derived from Chern-Simons

$$B = \nabla\times\mathbf{A} = -\frac{e}{\kappa}\rho$$

we have

$$\nabla^2\ln\rho = \pm 2\frac{e^2}{\kappa}\rho$$

which is the Liouville equation.

Formulation in terms of a curvature

SD is a geometrico-algebraic property of a fiber space : a differential form is equal to its Hodge dual.

For this model there is no clear geometric structure. However:

Define the two "potential-like" fields

$$\begin{aligned}\mathcal{A}_+ &= A_+ - \lambda\phi \\ \mathcal{A}_- &= A_- + \lambda\phi^\dagger\end{aligned}$$

and calculate the "curvature-like" fields

$$K_\pm \equiv \partial_\pm \mathcal{A}_\mp - \partial_\mp \mathcal{A}_\pm + [\mathcal{A}_\pm, \mathcal{A}_\mp]$$

We then have

$$\begin{aligned} & \operatorname{tr}\{K_+K_-\} \\ = \; & -2\left[\left(\partial_+ a^* + \partial_- a\right) + \lambda^2\left(\rho_1 - \rho_2\right)\right]^2 \\ & -\lambda^2\left|\left(\partial_+\phi_2^* + \partial_-\phi_1\right) + 2\left(a\phi_2^* - a^*\phi_1\right)\right|^2 \end{aligned}$$

or

$$-\operatorname{tr}\{K_+K_-\} \geq 0$$

since it is a sum of squares and the equality with zero is precisely the SD equations.

The self-duality indeed appears as a condition of a flat connection. A non-zero curvature means that the Euler fluid is *not* at stationarity.

The energy close to stationarity (or: self-duality)

We can use the expression of the energy, after applying the

Bogomolnyi procedure,

$$E = \frac{1}{2m} \mathrm{tr} \left((D_- \phi)^\dagger (D_- \phi) \right)$$

The energy becomes

$$E = \frac{1}{2m} \left(\rho_1 \left| \frac{1}{2\rho_1} \frac{\partial \rho_1}{\partial x_-} + i \frac{\partial \chi}{\partial x_-} - 2a^* \right|^2 + \rho_2 \left| \frac{1}{2\rho_2} \frac{\partial \rho_2}{\partial x_-} + i \frac{\partial \eta}{\partial x_-} + 2a^* \right|^2 \right)$$

and, if we take

$$\begin{aligned} \rho_1 &= \frac{1}{\rho_2} = \rho = \exp(\psi) \\ \chi &= -\eta \end{aligned}$$

we have

$$E = \frac{1}{2m}\left[\exp\left(\psi\right) + \exp\left(-\psi\right)\right] \left| \frac{1}{2}\frac{\partial \psi}{\partial x_-} + i\frac{\partial \chi}{\partial x_-} - 2a^* \right|^2$$

This form of the energy shows in what consists the approach to the stationarity and the formation of structure:

1. a constant ψ on the equilines combines its radial variation with that of of the angle χ;
2. the potentials a and a^* become velocities and they contain the derivatives along the equilines of the angle χ.

The expression of the FT current

The formula for the FT current

$$
\begin{aligned}
J^0 &= \left[\Psi^\dagger, \Psi\right] \\
J^i &= -\frac{i}{2}\left(\left[\Psi^\dagger, D_i\Psi\right] - \left[(D_i\Psi)^\dagger, \Psi\right]\right)
\end{aligned}
$$

We have

$$
\begin{aligned}
J^x &= \frac{1}{2}\left[2i(a-a^*)(\rho_1+\rho_2) - i\frac{\partial}{\partial x}(\rho_1-\rho_2)\right] H \\
J^y &= \frac{1}{2}\left[2(a+a^*)(\rho_1+\rho_2) - i\frac{\partial}{\partial y}(\rho_1-\rho_2)\right] H \\
J^0 &= (\rho_1-\rho_2)\, H
\end{aligned}
$$

or

$$J_{+} = \frac{1}{2} i \left(\rho_1 + \rho_2\right) \partial_{+} \left[\psi - \left(2i\chi\right)\right] - \frac{1}{2} i \partial_{+} \left(\rho_1 - \rho_2\right)$$

$$J_{-} = -\frac{1}{2} i \left(\rho_1 + \rho_2\right) \partial_{-} \left[\psi + \left(2i\chi\right)\right] - \frac{1}{2} i \partial_{-} \left(\rho_1 - \rho_2\right)$$

at SELF-DUALITY we have

$$\omega = -\sinh\psi$$

and it results

$$\begin{aligned} J_{+} &= \frac{1}{2} i \left(\rho_1 + \rho_2\right) \partial_{+} \left[\psi - \left(2i\chi\right)\right] - \frac{1}{2} i \partial_{+} \omega \\ J_{-} &= -\frac{1}{2} i \left(\rho_1 + \rho_2\right) \partial_{-} \left[\psi + \left(2i\chi\right)\right] - \frac{1}{2} i \partial_{-} \omega \end{aligned}$$

Is-there any pinch of vorticity?

The equations of motion of the FT model

The equation resulting from E_+.

$$\begin{aligned} & i\frac{\partial \phi_1}{\partial t} - 2ib\phi_1 \qquad (30) \\ = \quad & -\frac{1}{2}\frac{\partial^2 \phi_1}{\partial x^2} + \frac{1}{2}\left[\frac{\partial\left(a-a^*\right)}{\partial x}\phi_2 + \left(a-a^*\right)\frac{\partial \phi_2}{\partial x}\right] \\ & -\frac{1}{2}\frac{\partial \phi_1}{\partial x}\left(a-a^*\right) - \frac{1}{2}\left(a-a^*\right)^2\phi_1 \\ & -\frac{1}{2}\frac{\partial^2 \phi_2}{\partial y^2} + \frac{1}{2i}\left[\frac{\partial\left(a+a^*\right)}{\partial y}\phi_2 + \left(a+a^*\right)\frac{\partial \phi_2}{\partial y}\right] \\ & -\frac{1}{2}\frac{\partial \phi_2}{\partial y}\left(-\frac{1}{i}\right)\left(a+a^*\right) + \frac{1}{2}\left(a+a^*\right)^2\phi_2 \\ & -\left(\rho_1-\rho_2\right)\phi_1 \end{aligned}$$

The equation resulting from E_-.

$$
\begin{aligned}
& i\frac{\partial \phi_2}{\partial t} + 2ib\phi_2 \qquad (31) \\
= \quad & -\frac{1}{2}\frac{\partial^2 \phi_2}{\partial x^2} + \frac{1}{2}\left[\frac{\partial (a-a^*)}{\partial x}\phi_2 + (a-a^*)\frac{\partial \phi_2}{\partial x}\right] \\
& -\frac{1}{2}\frac{\partial \phi_2}{\partial x}(a-a^*) + \frac{1}{2}(a-a^*)^2\phi_2 \\
& -\frac{1}{2}\frac{\partial^2 \phi_2}{\partial y^2} + \frac{1}{2i}\left[\frac{\partial (a+a^*)}{\partial y}\phi_2 + (a+a^*)\frac{\partial \phi_2}{\partial y}\right] \\
& +\frac{1}{2i}\frac{\partial \phi_2}{\partial y}(a+a^*) + \frac{1}{2}(a+a^*)^2\phi_2 \\
& +(\rho_1-\rho_2)\phi_2
\end{aligned}
$$

Compare with Liouville (non-Abelian) case. Where is the dynamics?

Abelian-dominated dynamics

The last Lagrangian

In certain cases the model collapses to an Abelian structure, where (ϕ, A^{μ}) are complex scalar functions

$$\mathcal{L} = (D^{\mu}\phi)^{*} (D_{\mu}\phi) + \frac{1}{4}\kappa\varepsilon^{\mu\nu\rho} A_{\mu} F_{\nu\rho} - V\left(|\phi|^{2}\right)$$

where

$$D_{\mu}\phi = \frac{\partial\phi}{\partial x^{\mu}} + ieA_{\mu}\phi$$

and

$$V\left(|\phi|^{2}\right) = \frac{e^{2}}{\kappa^{2}} |\phi|^{2} \left(|\phi|^{2} - v^{2}\right)^{2}$$

with metric

$$g^{\mu\nu} = (1, -1, -1)$$

The equations of motion

$$D^\mu D_\mu \phi = -\frac{\partial V}{\partial \phi^*}$$

$$\frac{1}{2}\varepsilon^{\mu\nu\rho} F_{\nu\rho} = J^\rho$$

where

$$J^\mu = ie\left[\phi^* \left(D^\mu \phi\right) - \left(D^\mu \phi\right)^* \phi\right]$$

From the second equation of motion $B = -\frac{e}{\kappa}\rho$ one finds

$$A^0 = \frac{\kappa}{2e^2}\frac{B}{|\phi|^2} - \frac{1}{e}\frac{\partial}{\partial t}\left[\text{phase of } (\phi)\right]$$

In a field theory one can obtain the energy-momentum tensor by writing the action with the explicit presence of the metric $g^{\mu\nu}$

followed by variation of the action to this metric.

$$\begin{aligned} T_{\mu\nu} &= (D_\mu\phi)^* (D_\nu\phi) + (D_\mu\phi)(D_\nu\phi)^* \\ &\quad - g_{\mu\nu}\left[(D_\lambda\phi)^*(D_\lambda\phi) - V\left(|\phi|^2\right)\right] \end{aligned}$$

The energy is the *time-time* (00) component of this tensor

$$\begin{aligned} E &= \int d^2r \left[(D_0\phi)^*(D_0\phi) + (D_k\phi)^*(D_k\phi) + V\left(|\phi|^2\right)\right] \\ &= \int d^2r \left[\left(\frac{\partial|\phi|}{\partial t}\right)^2 + \frac{\kappa^2}{4e^2}\frac{B}{|\phi|^2} + (D_k\phi)^*(D_k\phi) + V\left(|\phi|^2\right)\right] \end{aligned}$$

The second term imposes that B and $|\phi|^2$ vanish in the same points. Then the magnetic flux lies in a ring around the zeros of $|\phi|^2$.

The *SELF-DUALITY*

The energy is transformed similar to the Bogomolnyi form

$$
\begin{aligned}
E &= \int d^2 r \left[\left| \left(D_x \pm i D_y \right) \phi \right|^2 \right. \\
&\quad + \left| \frac{\kappa}{2e} \phi^{-1} B \pm \frac{e^2}{\kappa} \phi^* \left(|\phi|^2 - v^2 \right) \right|^2 + \left. \left(\frac{\partial |\phi|}{\partial t} \right)^2 \right] \\
&\quad \pm e v^2 \Phi + \frac{1}{2} \int_{r=\infty} \mathbf{dl} \cdot \mathbf{J}
\end{aligned}
$$

Restrict to the states

1. static ($\partial / \partial t \equiv 0$);
2. the current goes to zero at infinity such that the last integral is zero.

Then the energy consists of a sum of squared terms plus an additional term that has a *topological* nature, proportional with the total magnetic flux through the area.

Taking to zero the squared terms we get

$$\begin{aligned}(D_x \pm i D_y)\,\phi &= 0 \\ eB &= \mp \frac{m^2}{2}\frac{|\phi|^2}{v^2}\left(1-\frac{|\phi|^2}{v^2}\right)\end{aligned}$$

The mass parameter is

$$m \equiv 2e^2 \frac{v^2}{\kappa}$$

These are the equations of self-duality and the energy in this case is *bounded from below* by the flux

$$E \geq e v^2 \,|\Phi|$$

The equation for the *ring-type* vortex

The first of the two SD equations can be written

$$eA^k = \pm\varepsilon^{kj}\partial_j \ln|\phi| + \partial^k \left[\text{phase of } \phi\right]$$

Replacing the potential in the second SD equation we get

$$\Delta \ln\left(|\phi|^2\right) - m^2 \frac{|\phi|^2}{v^2}\left(\frac{|\phi|^2}{v^2} - 1\right) = 0$$

equation that is valid in points where $|\phi| \neq 0$. For these points there is an additional term, a Dirac δ coming from taking the rotational operator applied on the term containing the phase of ϕ.

$$\Delta\psi = \exp(\psi)\left[\exp(\psi) - 1\right] + 4\pi \sum_{j=1}^{N} \delta\left(\mathbf{x} - \mathbf{x}_j\right)$$

The return of the topological constraint

At infinity ($|\phi| \simeq v$) the covariant derivative term goes to 0

$$D^k \phi \to 0 \text{ at } r \to \infty \qquad \partial_k \phi + ieA_k \phi \to 0$$

$$\int_{r=\infty} \mathbf{dl} \cdot \nabla \ln(\phi) = i \int d\,(\text{phase of } \phi) = 2\pi i n \tag{32}$$

The flux is

$$\Phi = \int d^2 r \,(\nabla \times \mathbf{A}) = \frac{2\pi}{e} n$$

The magnetic flux is discrete, *integer* multiple of a physical quantity. The topological constraint is ensured by a mapping from the circle at infinity into the circle representing the space of the internal phase of the field ϕ in the asymptotic region, $S^1 \to S^1$ classified according to the first homotopy group,

$$\pi_1 \left(S^1\right) = \mathbf{Z}$$

Why we substitute ρ *with* $\exp(\psi)$

The paper on **Bosonization of three dimensional non-abelian fermion field theories** by **Bralic, Fradkin, Schaposnik**.

The initial self-interacting massive fermionic $SU(N)$ theory in Euclidean $2+1=3$ space

$$\mathcal{L} = \overline{\psi}\left(i\partial\!\!\!/ + m\right)\psi - \frac{g^2}{2} j^{a\mu} j^a_\mu$$

NOTE

This is precisely the Lagrangian for the *Thirring* model, for which it is possible to demonstrate the **quantum** equivalence with the *sine-Gordon* model. See **Ketov**.

The model is here *Abelian*.

The action is

$$I_T[\psi] = \int d^2x \left[\overline{\psi}\gamma^\mu \partial_\mu \psi - m_F \overline{\psi}\psi - \frac{g}{2}\left(\overline{\psi}\gamma^\mu\psi\right)^2\right]$$

In order to show the equivalence the following substitution is made

$$\psi_{\pm} = \exp\left\{\frac{2\pi}{i\beta}\int_{-\infty}^{x} dx' \frac{\partial\phi\left(x'\right)}{\partial t} \mp \frac{i\beta}{2}\phi\left(x\right)\right\}$$

where

$$\psi \equiv \begin{pmatrix} \psi_{+} \\ \psi_{-} \end{pmatrix}$$

Note that ψ are spinors and ϕ are bosons.

The **equivalence** will now consist of the following statement:

The functions $\psi_{\pm}$ satisfy the Thirring equations of motion provided the function ϕ satisfies the sine-Gordon equation.

And viceversa.

This allows to demonstrate the equivalence between the correlation functions of the two models.

Between the coupling constant of the two theories there is the following relation

$$\frac{\beta^2}{4\pi} = \frac{1}{1 + g/\pi}$$

which shows that the strong coupling of the *Thirring* (fermions) model is mapped onto the weak coupling of the *sine-Gordon* (kinks and anti-kinks) model.

The mesons of the SG theory are the fermion-antifermion bound states of the Thirring theory.

The quantum bosonisation is done on the basis of the substitution shown above, but taking the *normal-ordered* form of the exponential.

$$\psi_{\pm} = C_{\pm} : \exp\left[A_{\pm}\left(x\right)\right] :$$

where

$$A_{\pm}\left(x\right) = \frac{2\pi m}{i\sqrt{\lambda}}\left(\int_{-\infty}^{x} dx' \frac{\partial \phi\left(x'\right)}{\partial t}\right) \mp \frac{i\sqrt{\lambda}}{2m}\phi\left(x\right)$$

This implies the relations

$$\begin{aligned} \frac{m_0^2 m^2}{\lambda} \cos\left(\frac{\sqrt{\lambda}}{m}\phi\right) &= -m_F \overline{\psi}\psi \\ -\frac{\sqrt{\lambda}}{2\pi m}\varepsilon^{\mu\nu}\partial_\nu \phi &= \overline{\psi}\gamma^\mu \psi \end{aligned}$$

We make the following **Remark**: We see that the density of spinors (or point-like vortices) $\overline{\psi}\psi$ is expressed as the cos function of the scalar field of the SG model. This looks very similar to what we have in our, more complex, model. In our model the density of vorticity (which represents the continuum limit of the density of point-like vortices) is

$$\phi^\dagger \phi = \rho_1 - \rho_2$$

and the two functions are

$$\begin{aligned} \rho_1 &\equiv \left|\phi_+\right|^2 \\ \rho_2 &\equiv \left|\phi_-\right|^2 \end{aligned}$$

We can introduce scalar streamfunctions for each of these densities, since they are associated with a sign of helicity

$$\rho_{1,2} = \exp(\psi_{1,2})$$

Then the total density of vorticity should be written

$$\begin{aligned}\phi^\dagger \phi &= \rho_1 - \rho_2 \\ &= \exp(\psi_1) - \exp(\psi_2)\end{aligned}$$

But we know that at self-duality

$$\Delta \ln \rho_1 + \Delta \ln \rho_2 = 0$$

or

$$\Delta \psi_1 + \Delta \psi_2 = 0$$

If we do not consider any background flow, then one possible solution of this equation is

$$\psi_1 = -\psi_2$$

and this gives the form of the density of vorticity

$$\begin{aligned} \phi^\dagger \phi &= \exp(\psi_1) - \exp(\psi_2) \\ &= 2 \sinh \psi \end{aligned}$$

We conclude that our theory is an extended form of the equivalence between the fermion system in plane (like the *Thirring* model) and the *Sinh-Gordon* model in plane.

Then, using the equivalences shown in the *Thirring-sine-Gordon* case, we can identify the function ϕ from their equation (the *sine-Gordon* variable) with the streamfunction ψ of our fluid, but multiplied with i.

And the current of fermions in their case $\overline{\psi}\gamma^\mu\psi$, which is proved to be expressed as a rotational of the SG function ϕ, appears in our case as follows: the current of point-like vortices is equal with the velocity since their ϕ is our streamfunction ψ and their rotational of the SG's ϕ is our rotational of ψ, or the physical velocity.

We can say that we assist at a typical scenario of equivalence between the system of point-like vortices and the system of *sinh-Gordon* streamfunction field, in a more extended, including *Non-Abelian* form.

The simplified result of the classical equivalence: *Thirring/sine-Gordon* was that the density of vorticity is *cos* of a bosonic field.

We do not need the bosonization, *i.e.* the substitution of the fermionic variable with the exponential of the bosonic variable. However *this can be a demonstration of the adequacy of the substitution*

$$\rho \equiv \exp(\psi)$$

we do at the end of the calculation: we do that since we have in mind the equivalence Thirring/sine-Gordon and the possibility to interpret our introduction of the streamfunction ψ as a similar relationship between the fermionic and bosonic fields.